More Resource for Game Comonads

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The Classic Modal Comonad

$\mathbb{M}_k(\mathfrak{A}, a_0)$ has[1]:

- Universe non-empty lists of max length $k + 1$ of the form $[a_0, a_1, \ldots, a_n]$ where $a_i \to a_{i+1}$.
- Distinguished element $[a_0]$.
- Edges between successors, e.g $[a_0, a_1] \to [a_0, a_1, a_2]$.
- Counit $\varepsilon[a_0, a_1, \ldots, a_n] = a_n$.
- Morphism $f : \mathbb{M}_k(\mathfrak{A}, a_0) \to (\mathfrak{B}, b_0)$ has coextension $f^* : \mathbb{M}_k(\mathfrak{A}, a_0) \to \mathbb{M}_k(\mathfrak{B}, b_0)$ given by:

$$
\begin{aligned}
&f^*[a_0, a_1, \ldots, a_n] \\
&= [f[a_0], f[a_0, a_1], \\
&\ldots, f[a_0, a_1, \ldots, a_n]]
\end{aligned}
$$

We also have $\mathbb{M}(\mathfrak{A}, a_0)$ with no bounds on the list length.

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[1] Signature has a single edge relation and no proposition symbols

Connections to Bisimulation Games

- Kleisli morphisms $(\mathfrak{A}, a_0) \to (\mathfrak{B}, b_0)$ correspond to winning strategies for duplicator in the k-step simulation game.
- Kleisli isomorphisms $(\mathfrak{A}, a_0) \cong (\mathfrak{B}, b_0)$ corresponds to winning strategies for duplicator in the k-step graded bisimulation game.
- Spans in the Eilenberg–Moore category of open pathwise embeddings $R(\mathfrak{A}, a_0) \leftarrow (\mathfrak{R}, r_0) \to R(\mathfrak{B}, b_0)$ correspond to winning strategies for duplicator in the k-step bisimulation game.

Using $\mathbb{M}$ without the resource bound gives us similar statements for full bisimulation and simulation.

The Coloured Modal Comonad

For a pointed set of colours (C, c_0), we can define a coloured modal comonad $\mathbb{M}_{(C,c_0)}(\mathfrak{A}, a_0)$ with:

- Universe non-empty lists of the form $[(a_0, c_0), (a_1, c_1), \ldots, (a_n, c_n)]$ where $a_i \to a_{i+1}$.
- Distinguished element $[(a_0, c_0)]$.
- Edges between successors, e.g. $[(a_0, c_0), (a_1, c_1)] \to [(a_0, c_0), (a_1, c_1), (a_2, c_2)]$.
- Counit $\varepsilon[(a_0, c_0), (a_1, c_1), \ldots, (a_n, c_n)] = a_n$.
- Morphism $f : \mathbb{M}_{(C,c_0)}(\mathfrak{A}, a_0) \to (\mathfrak{B}, b_0)$ has coextension $f^* : \mathbb{M}_{(C,c_0)}(\mathfrak{A}, a_0) \to \mathbb{M}_k(\mathfrak{B}, b_0)$ given by:

$$
\begin{aligned}
&f^*[(a_0, c_0), (a_1, c_1), \ldots, (a_n, c_n)] \\
&= [(f[(a_0, c_0)], c_0), (f[(a_0, c_0), (a_1, c_1)], c_1), \\
&\ldots, (f[(a_0, c_0), (a_1, c_1), \ldots, (a_n, c_n)], c_n)]
\end{aligned}
$$

Back to where we started

Now with explicit resources

Recovering $\mathbb{M}_k$ and $\mathbb{M}$

If choose our colours to be $(C, c_0) = (\{0, \ldots, k\}, 0)$, we recover $\mathbb{M}_k$ as a subcomonad by restricting to *ascending* sequences:

$$[(a_0, 0), (a_1, 1), \ldots, (a_n, n)]$$

with $n \leq k$. We can recover $\mathbb{M}$ via a similar restriction.

Back to where we started

Now with explicit resources

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Recovering $\mathbb{M}_k$ via a different choice of colours

We can also choose colours $(C, c_0) = (\{0, \dots, k\}, k)$ and recover $\mathbb{M}_k$ as a subcomonad by restricting to *descending* sequences:

$$[(a_0, k), (a_1, k-1), \dots, (a_n, k-n)]$$

Wasting resources

Instead of just allowing sequences of the form:

$$[(a_0, 5), (a_1, 4), (a_2, 3), (a_3, 2), (a_4, 1), (a_5, 0)]$$

we could generalize slightly, and allow spoiler to play *wastefully*, allowing sequences of the form:

$$[(a_0, 5), (a_1, \mathbf{4}), (a_2, \mathbf{2}), (a_3, 1), (a_4, 0)]$$

Intuitively, we allow spoiler to choose to expend more resources than necessary.

More resources

What happens if we start with the ordinal ω of resources? We have sequences of the form:

$$[(a_0, \omega), (a_1, 3), (a_2, 2), (a_3, 1), (a_4, 0)]$$

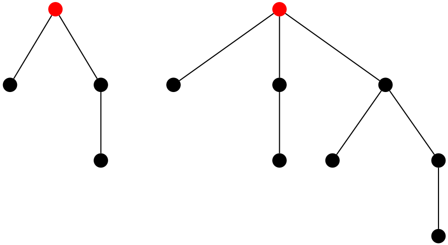
Spoiler *has to make a choice* about how long they want to play after their first move.

Difference to the unrestricted $\mathbb{M}$

This is not the same as $\mathbb{M}$, as all the plays above have finite length.

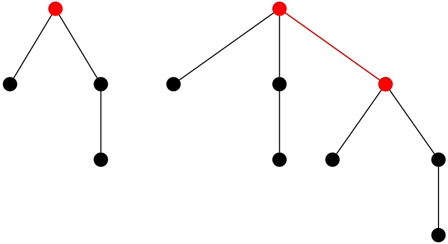
An example play

Resources $= \omega$, spoiler to play



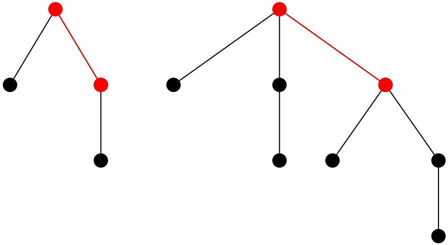
An example play

Resources = 2, duplicator to play



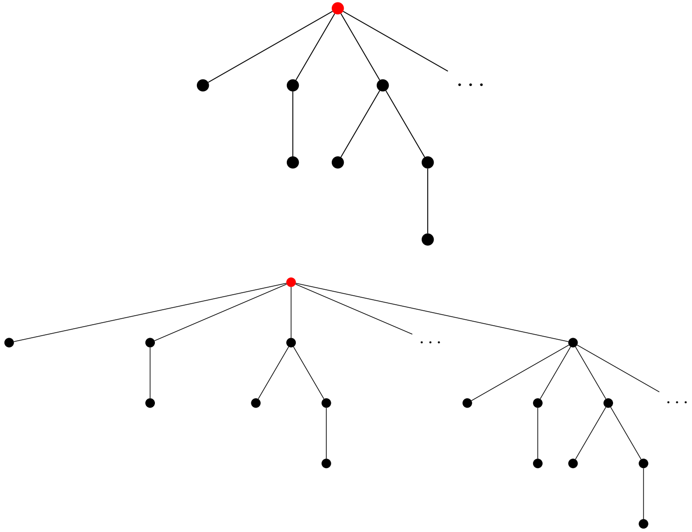
An example play

Resource = 2, spoiler to play



An example play on richer structures, insufficiency of ω

Resources $= \omega$, spoiler to play



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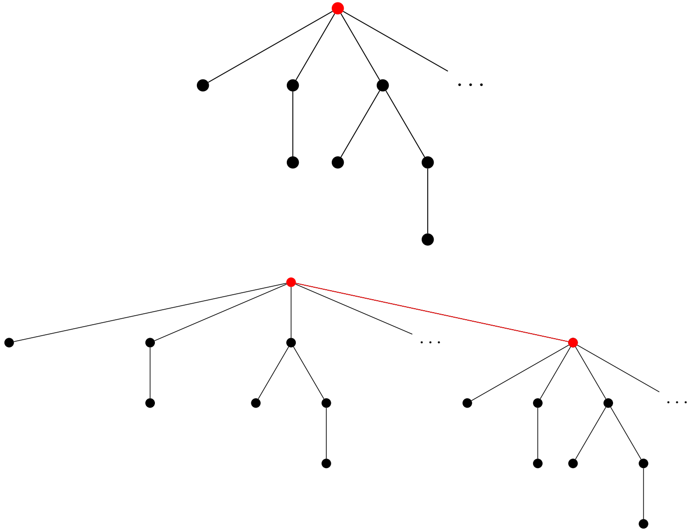
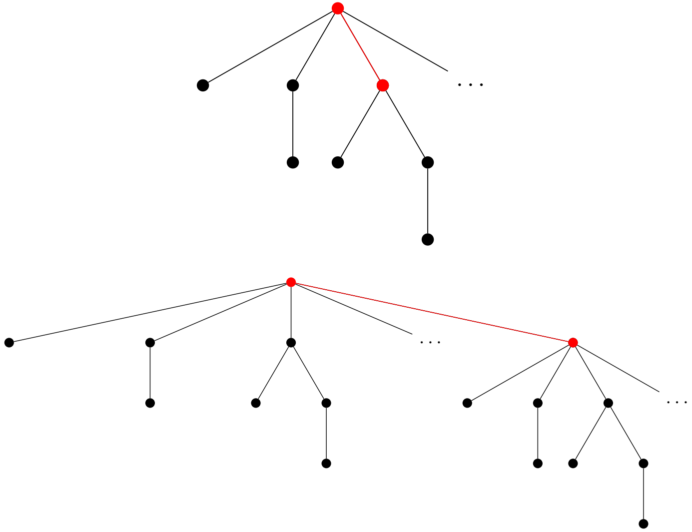
Resources = 2, duplicator to play



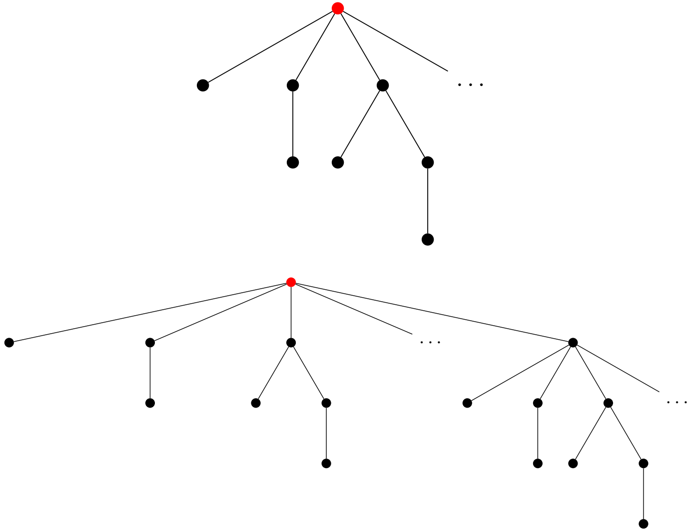
An example play on richer structures, insufficiency of ω

Resources = 2, spoiler to move



An example play on richer structures - sufficiency of $\omega + 1$

Resources $= \omega + 1$, spoiler to play



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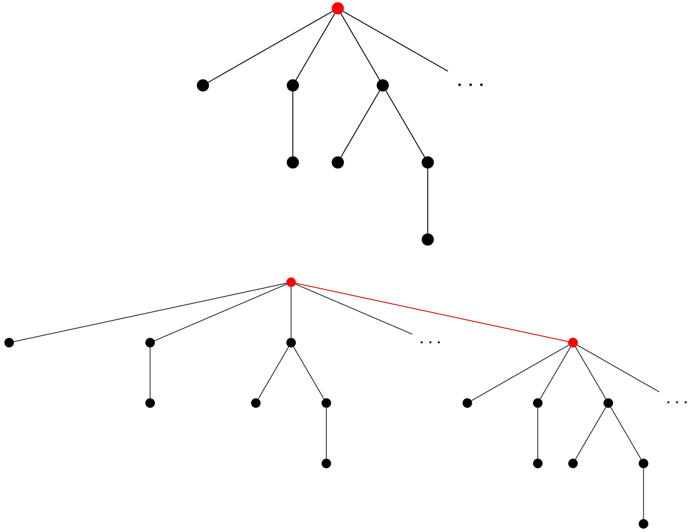

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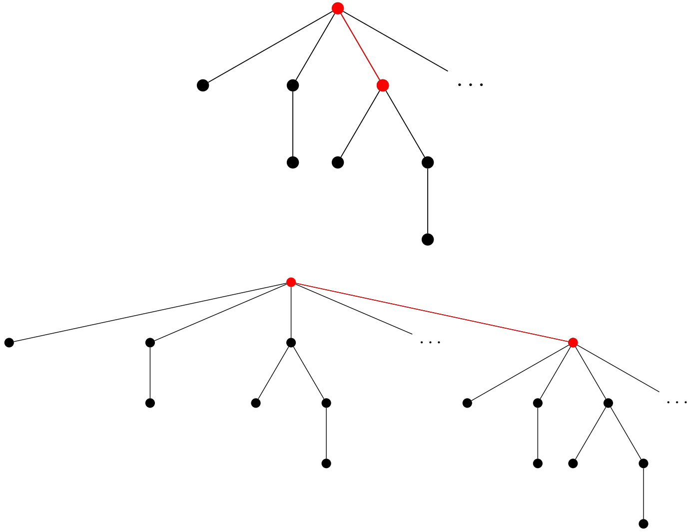
Resources $= \omega$, spoiler to move



What's going on logically?

Modal depth in infinitary modal logic:

$$\mathbf{MD}(\neg\varphi) = \mathbf{MD}(\varphi)$$
$$\mathbf{MD}(\bigvee_{i \in I}) = \sup\{\varphi_i \mid i \in I\}$$
$$\mathbf{MD}(\bigwedge_{i \in I}) = \sup\{\varphi_i \mid i \in I\}$$
$$\mathbf{MD}(\Box\varphi) = \mathbf{MD}(\varphi) + 1$$
$$\mathbf{MD}(\Diamond\varphi) = \mathbf{MD}(\varphi) + 1$$

Next steps

- We can adapt other comonads in a similar way, for example the Ehrenfeucht–Fraïssé comonad adapts to capture the dynamic Ehrenfeucht–Fraïssé game described in [Vää11] for infinitary logic.
- Lots of details and relationships remain to be checked - combinatorial parameters, relationships between the various comonads.
- Can the transfinite Ehrenfeucht–Fraïssé game for infinite quantifier logic be given a comonadic treatment?

References

Jouko Väänänen.
Models and games, volume 132 of *Cambridge studies in advanced mathematics*.
Cambridge University Press, 2011.