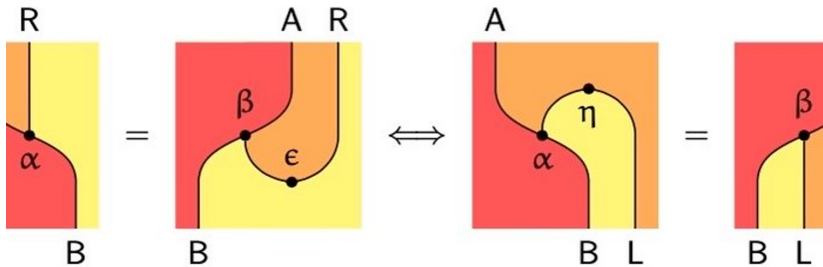


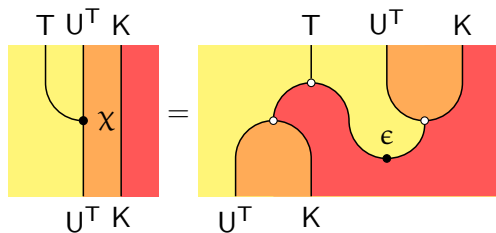
# The Graphical Theory of Monads

(Joint work with Ralf Hinze)

4-12-2023



# Introduction



Aims:

- ▶ Introduction to string diagrams.
- ▶ Examples drawn from monad theory.
- ▶ A larger example from the formal theory of monads.

# Category Theory in Pictures

Traditional notation

Categories

 $\mathcal{C}$ 

Functors

 $\mathcal{D} \xleftarrow{\quad \text{F} \quad} \mathcal{C}$ 

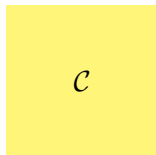
Natural Transformations

$$\begin{array}{ccc} & \text{F} & \\ \mathcal{D} & \begin{array}{c} \curvearrowleft \\ \Downarrow \alpha \\ \curvearrowright \end{array} & \mathcal{C} \\ & \text{G} & \end{array} \quad .$$

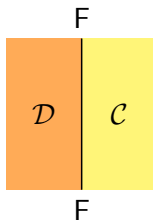
# Category Theory in Pictures

String diagram notation

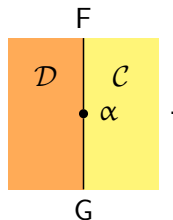
Categories



Functors



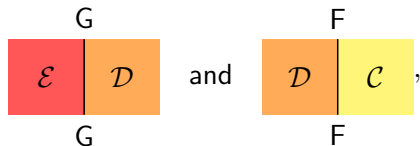
Natural Transformations



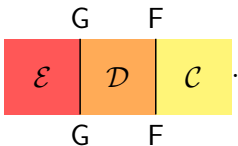
# Category Theory in Pictures

## Composition of functors

Given functors  $G : \mathcal{D} \rightarrow \mathcal{E}$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$ , in pictures,



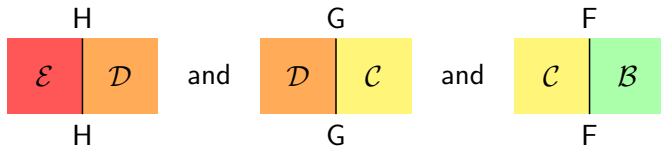
their composite  $G \circ F : \mathcal{C} \rightarrow \mathcal{E}$  is drawn as follows:



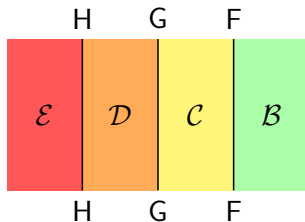
# Category Theory in Pictures

## Associativity of composition

Given three functors  $H : \mathcal{D} \rightarrow \mathcal{E}$ ,  $G : \mathcal{C} \rightarrow \mathcal{D}$  and  $F : \mathcal{B} \rightarrow \mathcal{C}$ :



the composites  $H \circ (G \circ F)$  and  $(H \circ G) \circ F$  are both depicted:



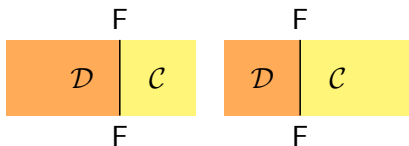
# Category Theory in Pictures

## Identity functor notation

We draw the identity functor  $\text{Id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  as the corresponding coloured region:



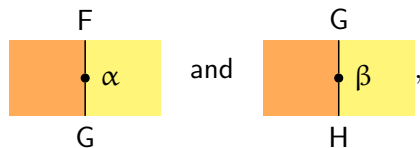
We then can draw  $\text{Id}_{\mathcal{D}} \circ F$  and  $F \circ \text{Id}_{\mathcal{C}}$  as follows:



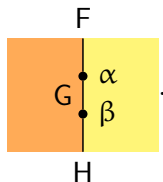
# Category Theory in Pictures

## Vertical composition of natural transformations

Given natural transformations,



we depict their vertical composite  $\beta \cdot \alpha : F \rightarrow H$  as the following diagram:

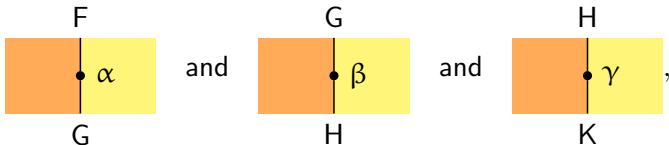




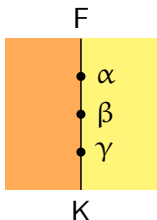
# Category Theory in Pictures

## Associativity of vertical composition

Given three natural transformations,



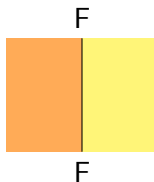
The vertical composites  $\gamma \cdot (\beta \cdot \alpha) : F \rightarrow K$  and  $(\gamma \cdot \beta) \cdot \alpha : F \rightarrow K$  are both drawn:



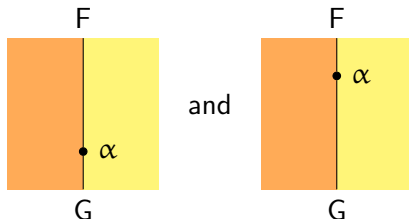
# Category Theory in Pictures

## Identity natural transformation notation

We draw the identity natural transformation  $id_F : F \rightarrow F : \mathcal{C} \rightarrow \mathcal{D}$  as the corresponding edge:



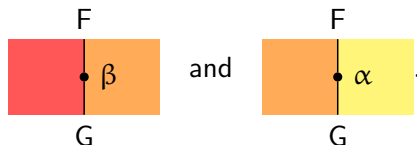
The two composites  $id_G \cdot \alpha$  and  $\alpha \cdot id_F$  are depicted:



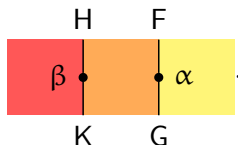
# Category Theory in Pictures

## Horizontal composition of natural transformations

Consider natural transformations:



We denote their horizontal composite  $\beta \circ \alpha : H \circ F \rightarrowtail K \circ G$  via horizontal diagrammatic composition:



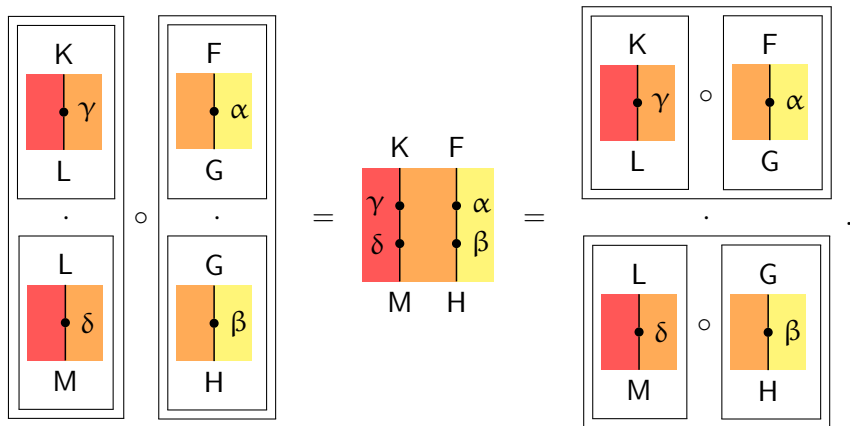
# Category Theory in Pictures

## The interchange law

The two forms of composition satisfy the **interchange law**:

$$(\delta \cdot \gamma) \circ (\beta \cdot \alpha) = (\delta \circ \beta) \cdot (\gamma \circ \alpha).$$

which again is built into the notation:



# Category Theory in Pictures

## Elevator equations

Using both vertical and horizontal composition, and our convention for drawing identity natural transformations, we obtain what Dubuc suggestively calls the **elevator equations**.

The diagram illustrates the elevator equations, which are identities involving the composition of natural transformations. It consists of three diagrams connected by equals signs, all set against a background of three vertical colored bands: red on the left, orange in the middle, and yellow on the right.

- Diagram 1 (Left):** Labeled with  $H$  and  $F$  at the top, and  $K$  and  $G$  at the bottom. It features two vertical black lines. The left line has a dot labeled  $\beta$  in the red band. The right line has a dot labeled  $\alpha$  in the orange band.
- Diagram 2 (Middle):** Labeled with  $H$  and  $F$  at the top, and  $K$  and  $G$  at the bottom. It features two vertical black lines. The left line has a dot labeled  $\beta$  in the red band. The right line has a dot labeled  $\alpha$  in the yellow band.
- Diagram 3 (Right):** Labeled with  $H$  and  $F$  at the top, and  $K$  and  $G$  at the bottom. It features two vertical black lines. The left line has a dot labeled  $\beta$  in the red band. The right line has a dot labeled  $\alpha$  in the yellow band.

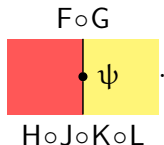
The three diagrams are connected by equals signs, representing the identity:

$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3}.$$

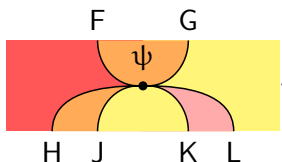
# Category Theory in Pictures

## Composite wires

We could draw the natural transformation  $\psi : F \circ G \rightarrow H \circ J \circ K \circ L$  as:



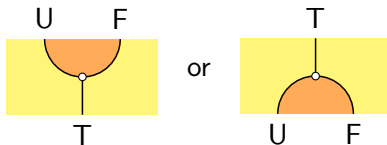
Instead, we draw separate wires for each element of the composite:



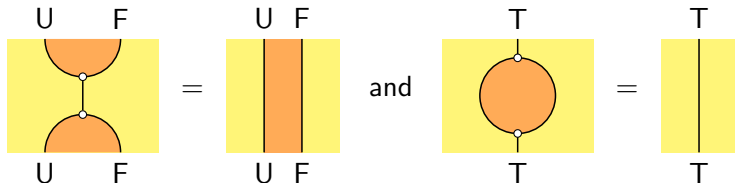
# Category Theory in Pictures

## Explicit identities and cancellation

If we have an equation between functors, such as  $T = U \circ F$ , we can exploit this in our diagrams using explicit identity vertices:



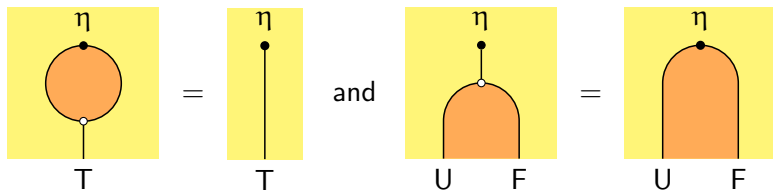
These satisfy obvious cancellation identities:



# Category Theory in Pictures

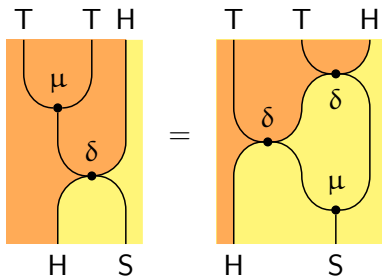
## Explicit identities and fusion

We can also fuse identities with other vertices in obvious ways, for example for  $\eta : \text{Id} \rightarrow T$ , with  $T = U \circ F$  as before:

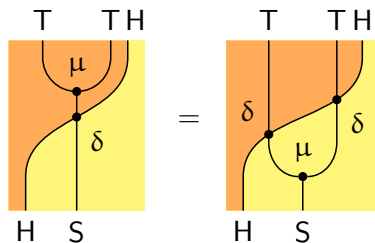




## Artistic choices matter - overdoing symmetry

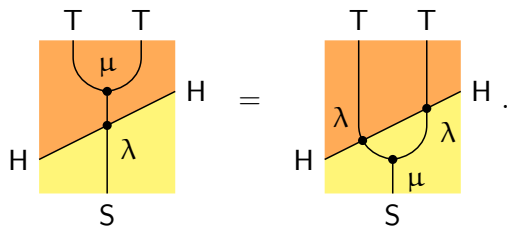


## Artistic choices matter - providing intuition



# Category Theory in Pictures

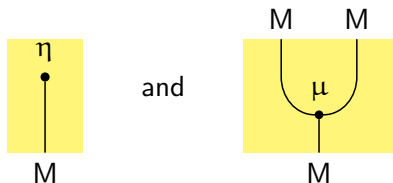
Artistic choices - stretching the notation



# Key Structures

## Monads

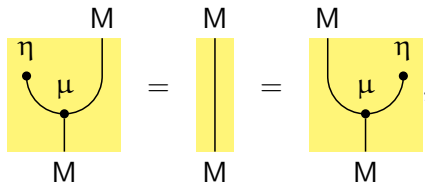
Our fundamental object of study is that of a monad. A **monad** on a category  $\mathcal{C}$  consists of an endofunctor  $M : \mathcal{C} \rightarrow \mathcal{C}$ , and **unit** and **multiplication** natural transformations



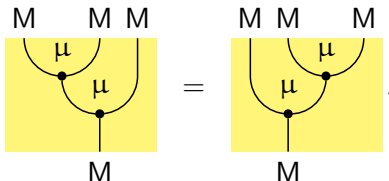
# Key Structures

## Monad axioms

The unit and multiplication are required to satisfy the following **unitality** and **associativity** equations:



The unitality equation is represented by a diagram showing three yellow rectangular boxes connected by equals signs. The first box contains a vertical line from the bottom to a dot, with a curved line labeled  $\mu$  connecting this dot to another dot on the left labeled  $\eta$ . The top of the box is labeled  $M$  and the bottom is labeled  $M$ . The second box is a single vertical line from bottom to top, with  $M$  at both ends. The third box is similar to the first but the dot on the left is labeled  $\eta$  and the dot on the right is labeled  $\mu$ . The top of the box is labeled  $M$  and the bottom is labeled  $M$ .

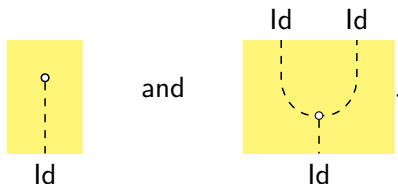


The associativity equation is represented by a diagram showing two yellow rectangular boxes connected by an equals sign. The left box has a vertical line from the bottom to a dot, with two curved lines labeled  $\mu$  connecting this dot to two dots above it. The top of the box is labeled  $M$  and the bottom is labeled  $M$ . The right box has a vertical line from the bottom to a dot, with two curved lines labeled  $\mu$  connecting this dot to two dots above it. The top of the box is labeled  $M$  and the bottom is labeled  $M$ .

# Key Structures

## Monads - Example

The identity functor on  $\mathcal{C}$  carries the structure of a monad:



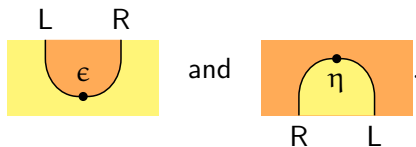
Verifying all the axioms boils down to confirming equations of the form:

$\square = \square$

# Key Structures

## Adjunctions

An **adjunction** between a pair of functors  $L : \mathcal{D} \rightarrow \mathcal{C}$  and  $R : \mathcal{C} \rightarrow \mathcal{D}$  consists of a pair of **counit** and **unit** natural transformations:

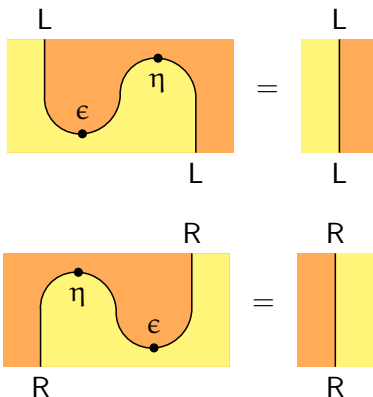


These are often referred to as a **cup** and **cap**.

# Key Structures

## Adjunction axioms

The unit and counit are required to satisfy the following **snake equations**



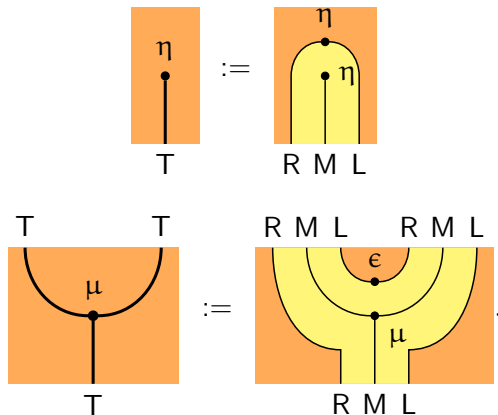
We will write  $L \dashv R : \mathcal{C} : \mathcal{C} \rightarrow \mathcal{D}$  to denote such an adjunction.  $L$  and  $R$  are referred to as **left** and **right** adjoints respectively.



# Key Structures

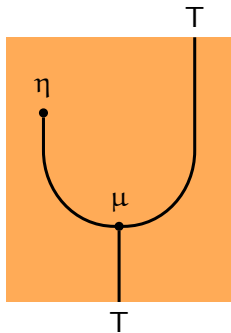
## Huber's construction

Given a monad  $(M : \mathcal{C} \rightarrow \mathcal{C}, \eta, \mu)$  and an adjunction  $L \dashv R : \mathcal{C} : \mathcal{D}$ , we can build a new monad on  $T = R \circ M \circ L$  as follows:

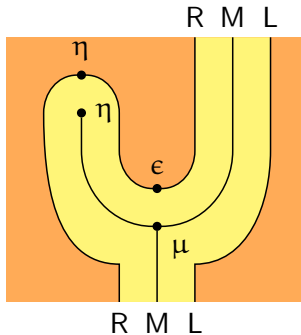


# Key Structures

Huber's construction - unit axiom

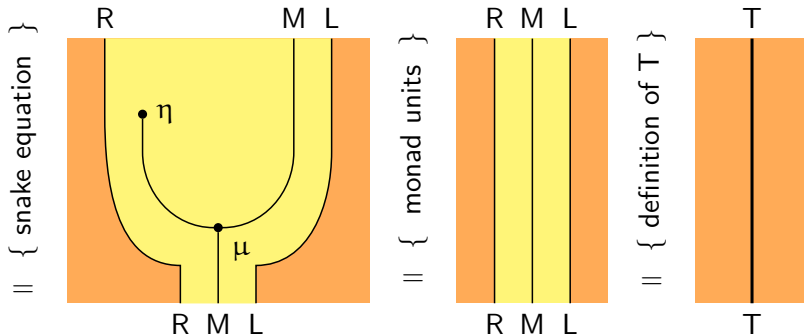


$\equiv$  { definitions }



# Key Structures

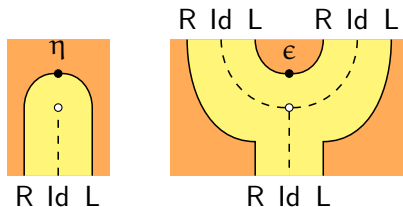
Huber's construction - unit axiom



The other unit and the multiplication axiom have similar proofs.

# Key Structures

As a special case of Huber's construction, every adjunction induces a monad:



# Abstraction and Formality

## 2-Categories

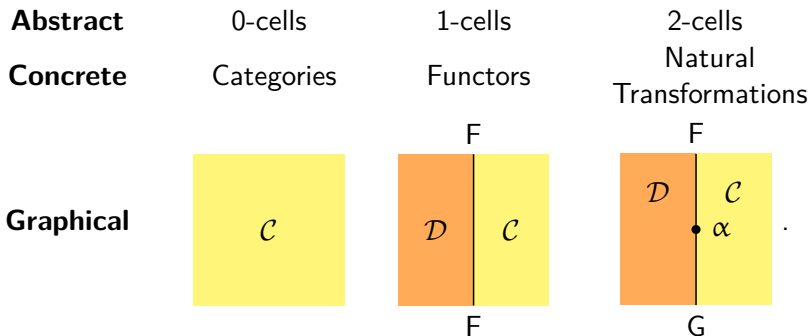
A **2-category** is an abstraction of the composition of categories, functors and natural transformations, in the same way we can think of categories as an abstraction of how functions between sets compose.

- ▶ We now speak of 0,1 and 2-cells.
- ▶ There are identity 1 and 2-cells.
- ▶ There are two notions of composition, horizontal and vertical.
- ▶ Composition and identities satisfy the same equations discussed earlier.

# Abstraction and Formality

String diagrams as notation for 2-categories

Our graphical notation extends to this abstract setting



# Abstraction and Formality

## Transferring notions to other settings

The definitions of monads and adjunctions transfer to the 2-categorical setting, and we can ask what they mean concretely in those settings. For example:

- ▶ There is a 2-category in which monads are preordered sets.
- ▶ There is a 2-category in which monads are internal categories.
- ▶ There is a 2-category in which monads are enriched categories.
- ▶ ...

# Abstraction and Formality

## Formal category theory

We can aim to “do category theory” within an arbitrary 2-category. This is referred to as doing **formal category theory**. Specifically for monads, many results work at this level of abstraction:

- ▶ Adjunctions induce monads.
- ▶ Monads can be composed using distributive laws.
- ▶ Codensity monads are induced by Kan extensions.
- ▶ ...



# Abstraction and Formality

## Monads and adjunctions

Key components of ordinary monad theory:

$$\begin{array}{ccccc} \mathcal{C}_T & \xrightarrow{J} & \mathcal{D} & \xrightarrow{K} & \mathcal{C}^T \\ F_T \uparrow \text{---} \downarrow U_T & & F \uparrow \text{---} \downarrow U & & F^T \uparrow \text{---} \downarrow U^T \\ C & & C & & C \end{array}$$

Without this, we cannot

- ▶ Show every monad arises from an adjunction.
- ▶ Show every monad is a codensity monad.
- ▶ Even talk about various lifting results.

# Abstraction and Formality

## The formal theory of monads

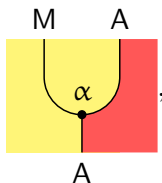
Street's formal theory of monads solves this problem:

- ▶ Elegant abstractions of the Eilenberg–Moore and Kleisli constructions are identified.
- ▶ The machinery looks rather advanced - auxiliary 2-categories are introduced, and results phrased in terms of the existence of various 2-adjoints.
- ▶ A surprisingly large amount of monad theory can be developed at this level of abstraction.

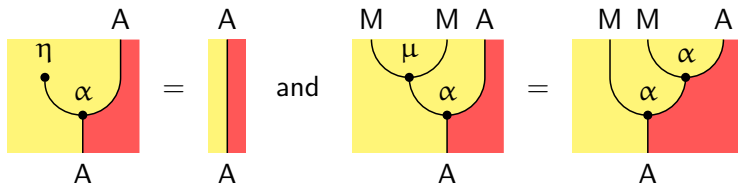
# Actions

## Left monad actions

Given a monad  $(M : \mathcal{C} \rightarrow \mathcal{C}, \eta, \mu)$ , a **left M-action** of  $M$  on  $A : \mathcal{E} \rightarrow \mathcal{C}$  is an  $\alpha : M \circ A \rightarrow A$ ,



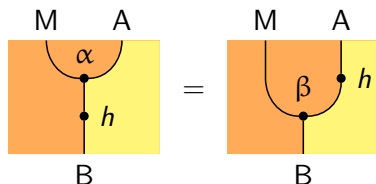
that respects the unit and multiplication, in that the following equations hold.



# Actions

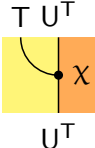
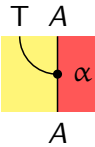
## Action transformations


Given a pair of actions of the *same* monad and from the *same* source, a **transformation of actions**, written  $h : (A, \alpha) \rightarrow (B, \beta)$ , is a  $h : A \rightarrow B$  such that the **right turn axiom** holds:



# Actions

## Universal left actions - one dimensional

We say that left T-action  is **universal** if for every 

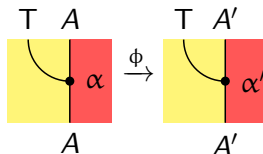
there exists a unique  such that:

$$\begin{array}{c} T \ U^T \ K \\ \text{[Diagram: Yellow rectangle, vertical orange line with dot } \chi, \text{ curved line from top left to } \chi, \text{ orange rectangle to the right of line, red rectangle to the right of orange rectangle]} \\ U^T \ K \end{array} = \begin{array}{c} T \ U^T \ K \\ \text{[Diagram: Yellow rectangle, vertical orange line with dot } \alpha, \text{ curved line from top left to } \alpha, \text{ orange semi-circles above and below } \alpha, \text{ orange rectangle to the right of line, red rectangle to the right of orange rectangle]} \\ U^T \ K \end{array},$$

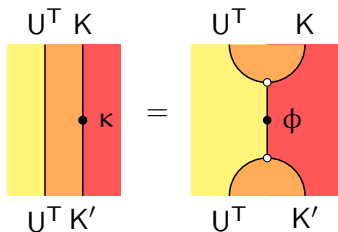
# Actions

Universal left actions - two dimensional

...and for every action transformation



there exists a unique  $\kappa : K \rightarrow K'$  such that





where  $K$  and  $K'$  are induced by the actions  $\alpha$  and  $\alpha'$  respectively.

# Actions


## Eilenberg–Moore objects

We shall say that  is an **Eilenberg–Moore object** for

monad  $T$  on  if there is a  carrying a universal left

$U^T$

$U^T$

T-action  .

$U^T$

Symbolically, we shall write  $\mathcal{C}^T$  for the Eilenberg–Moore object.

# Every monad arises from an adjunction

## The plan

For monad  $(T : \mathcal{C} \rightarrow \mathcal{C}, \eta, \mu)$ :

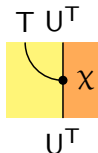
- ▶ Assume  $T$  has an Eilenberg–Moore object.
- ▶ Use the universal property to find suitable candidate left and right adjoints.
- ▶ Use the universal property to find putative units and counits.
- ▶ Show that this data yields an adjunction.
- ▶ Show that the adjunction induces the original monad via Huber's construction.



# Every monad arises from an adjunction

A candidate right adjoint

Assuming a universal left  $T$ -action



we need to find a candidate right adjoint of type  $\mathcal{C}^T \rightarrow \mathcal{C}$ . We already have a suitable candidate,  $U^T$ .

# Every monad arises from an adjunction

$\mu$  is a left T-action

To find a suitable left adjoint, we look for a left T-action that will induce a morphism of the right type. Via the two monad axioms:

$$\begin{array}{c} M \\ \eta \\ \mu \\ M \end{array} = \begin{array}{c} M \\ M \end{array},$$

$$\begin{array}{c} M \quad M \quad M \\ \mu \\ \mu \\ M \end{array} = \begin{array}{c} M \quad M \quad M \\ \mu \\ \mu \\ M \end{array}.$$

We observe that the multiplication is a left action.

# Every monad arises from an adjunction

$\mu$ -equation

The universal property tells us that  $\mu$  will induce  $F^T : \mathcal{C} \rightarrow \mathcal{C}^T$  such that the following key equation holds:

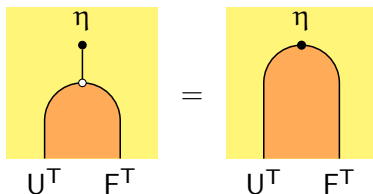
The diagram illustrates the  $\mu$ -equation in a string diagrammatic notation. It consists of two square boxes connected by an equals sign, both set against a yellow background. The left box contains a vertical orange line. A black line enters from the top left, curves to the right, and terminates at a black dot on the orange line, labeled with the Greek letter  $\chi$ . The top of the box is labeled  $T \ U^T \ F^T$  and the bottom is labeled  $U^T \ F^T$ . The right box contains a vertical line with two orange semi-circles at the top and bottom. The black line from the top left enters the top semi-circle, passes through a white dot, then through a black dot labeled  $\mu$ , and finally through another white dot at the bottom semi-circle. The top of the box is labeled  $T \ U^T \ F^T$  and the bottom is labeled  $U^T \ F^T$ .

Note in particular that  $T = U^T \circ F^T$ .

# Every monad arises from an adjunction

The inevitable choice of unit

Given the way Huber's construction worked, there is only one choice for the unit of the adjunction, which is to use the unit of the monad, exploiting the fact  $T = U^T \circ F^T$ .



# Every monad arises from an adjunction

Constructing a candidate counit

The left T-action axiom

$$\begin{array}{c} T \quad T \quad A \\ \text{[Diagram with } \mu \text{ and } \chi \text{]} \\ A \end{array} = \begin{array}{c} T \quad T \quad U^T \\ \text{[Diagram with } \chi \text{]} \\ U^T \end{array}$$

is equivalent to saying  $\chi$  is an action transformation of type:

$$\begin{array}{c} T \quad T \quad U^T \\ \text{[Diagram with } \mu \text{]} \\ T \quad U^T \end{array} \xrightarrow{\chi} \begin{array}{c} T \quad U^T \\ \text{[Diagram with } \chi \text{]} \\ U^T \end{array}$$

and so by the universal property induces a 2-cell



# Every monad arises from an adjunction

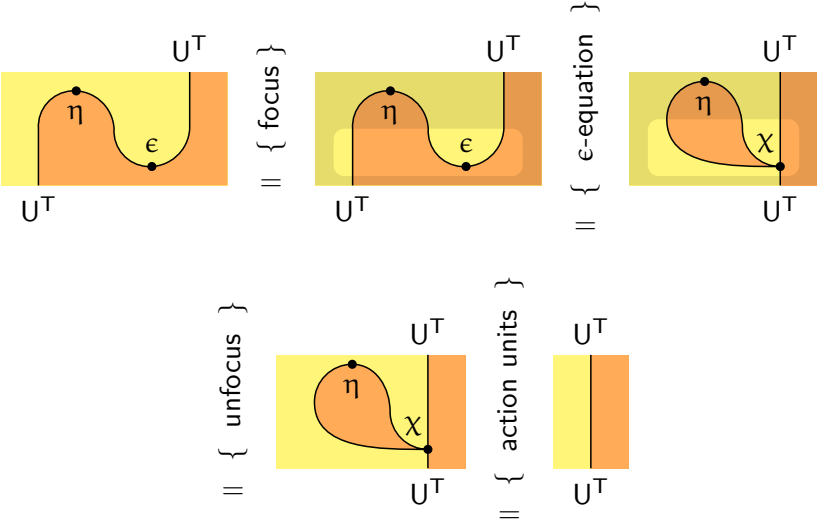
$\epsilon$ -equation

Furthermore, the U.P. implies following key equation holds:

$$\begin{array}{c} U^T F^T \quad U^T \\ \text{[Diagram: A yellow rectangle with an orange vertical strip on the right. A yellow semi-circle with a black dot at its center is on the orange strip. The semi-circle is labeled with the Greek letter epsilon (\epsilon).]} \\ U^T \end{array} = \begin{array}{c} U^T F^T U^T \\ \text{[Diagram: A yellow rectangle with an orange vertical strip on the right. A yellow semi-circle with a black dot at its center is on the orange strip. The semi-circle is labeled with the Greek letter chi (\chi).]} \\ U^T \end{array}$$

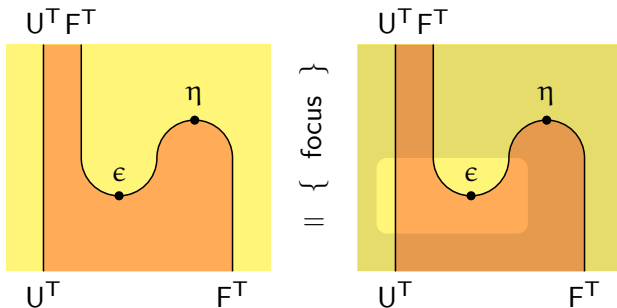
# Every monad arises from an adjunction

Proving the first snake equation



# Every monad arises from an adjunction

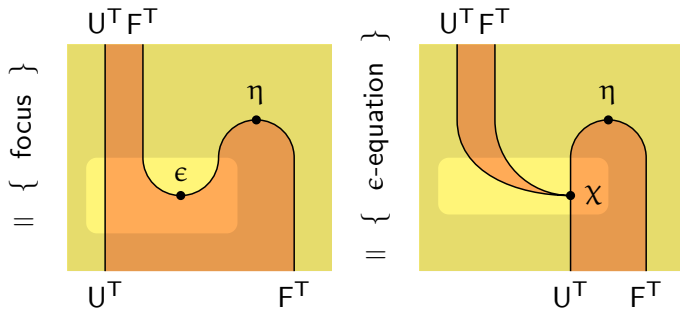
Proving the second snake equation





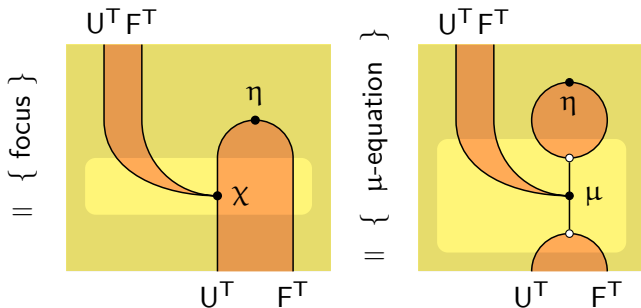
# Every monad arises from an adjunction

Proving the second snake equation



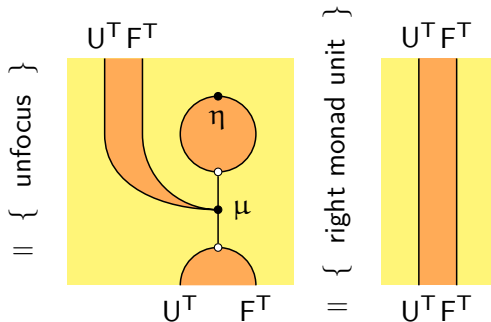
# Every monad arises from an adjunction

Proving the second snake equation



# Every monad arises from an adjunction

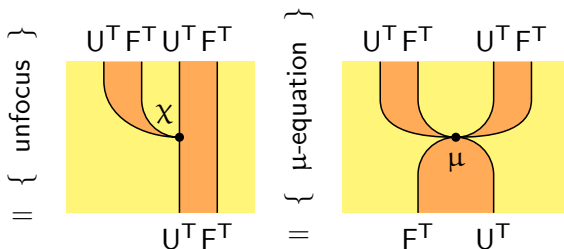
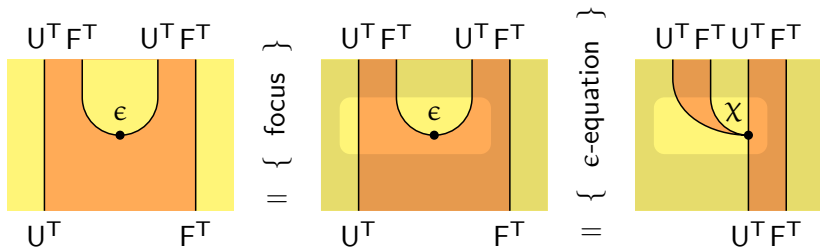
Proving the second snake equation



We then note by the U.P. the operation  $U^T \circ (-)$  is injective.

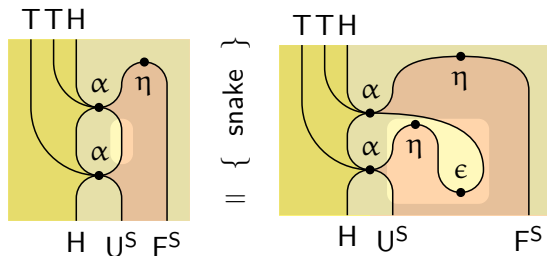
# Every monad arises from an adjunction

Confirming we recover the original monad



# Conclusion

Going further



- ▶ Other results have natural graphical arguments - the Eilenberg–Moore resolution is terminal, Eilenberg–Moore laws classify liftings, Beck distributive laws lift monads to Eilenberg–Moore objects, every monad is a codensity monad,...
- ▶ Duality gives results for Kleisli construction and comonads by “flipping pictures”