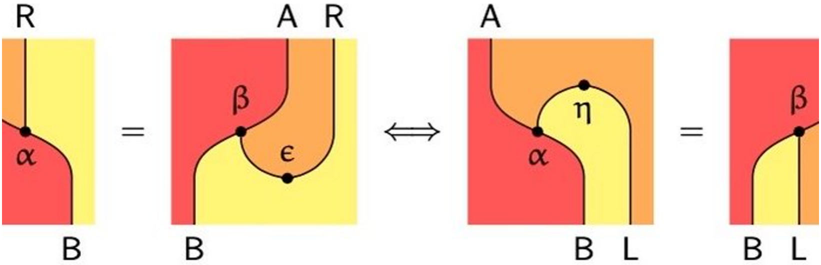
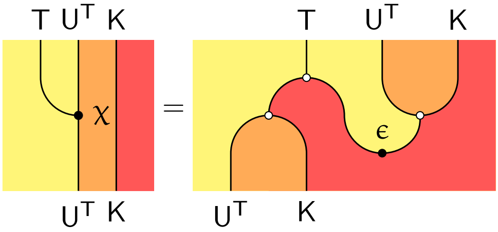
The Graphical Theory of Monads

(Joint work with Ralf Hinze)

4-12-2023



Introduction



Aims:

- Introduction to string diagrams.
- Examples drawn from monad theory.
- A larger example from the formal theory of monads.

Category Theory in Pictures

Traditional notation

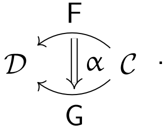
Categories

$\mathcal{C}$

Functors

$$\mathcal{D} \xleftarrow[F]{} \mathcal{C}$$

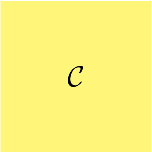
Natural Transformations



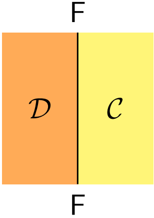
Category Theory in Pictures

String diagram notation

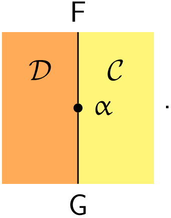
Categories



Functors



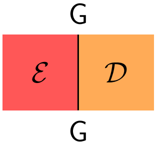
Natural Transformations



Category Theory in Pictures

Composition of functors

Given functors $\mathsf{G} : \mathcal{D} \to \mathcal{E}$ and $\mathsf{F} : \mathcal{C} \to \mathcal{D}$, in pictures,



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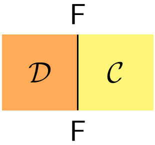

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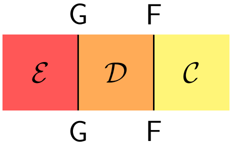
their composite $\mathsf{G} \circ \mathsf{F} : \mathcal{C} \to \mathcal{E}$ is drawn as follows:



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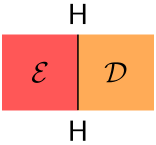
Category Theory in Pictures

Associativity of composition

Given three functors $\mathsf{H} : \mathcal{D} \to \mathcal{E}$, $\mathsf{G} : \mathcal{C} \to \mathcal{D}$ and $\mathsf{F} : \mathcal{B} \to \mathcal{C}$:



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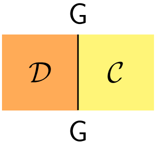

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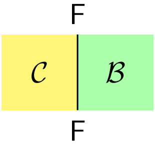
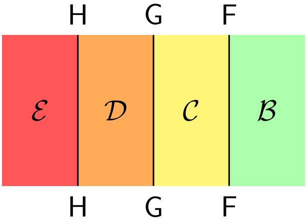


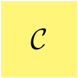
the composites H∘(G∘F) and (H∘G)∘F are both depicted:



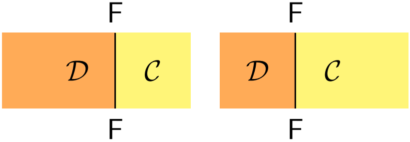
Category Theory in Pictures

Identity functor notation

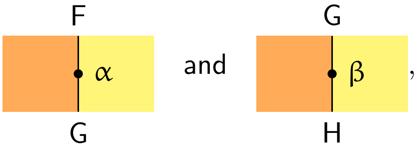
We draw the identity functor $\mathsf{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ as the corresponding coloured region:



We then can draw $\mathsf{Id}_{\mathcal{D}} \circ \mathsf{F}$ and $\mathsf{F} \circ \mathsf{Id}_{\mathcal{C}}$ as follows:



Given natural transformations,



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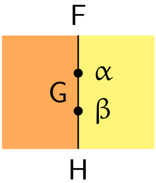
we depict their vertical composite $\beta \cdot \alpha : F \xrightarrow{\cdot} H$ as the following diagram:



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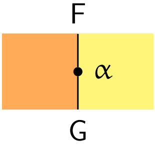
Associativity of vertical composition

Given three natural transformations,



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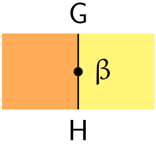

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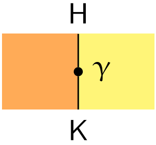

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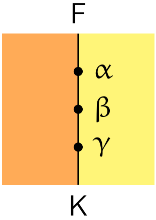
The vertical composites $\gamma \cdot (\beta \cdot \alpha) : \mathsf{F} \dot{\to} \mathsf{K}$ and $(\gamma \cdot \beta) \cdot \alpha : \mathsf{F} \dot{\to} \mathsf{K}$ are both drawn:



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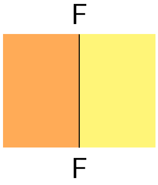
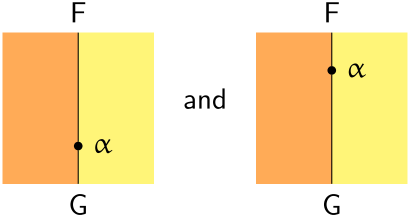
Category Theory in Pictures

Identity natural transformation notation

We draw the identity natural transformation $id_{\mathsf{F}} : \mathsf{F} \dot{\rightarrow} \mathsf{F} : \mathcal{C} \rightarrow \mathcal{D}$ as the corresponding edge:



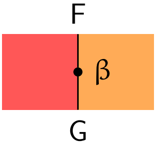
The two composites $id_{\mathsf{G}} \cdot \alpha$ and $\alpha \cdot id_{\mathsf{F}}$ are depicted:



Category Theory in Pictures

Horizontal composition of natural transformations

Consider natural transformations:



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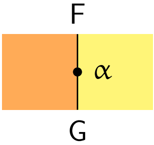

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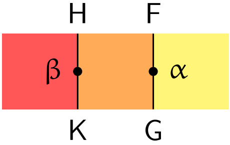
We denote their horizontal composite $\beta \circ \alpha : H \circ F \xrightarrow{\cdot} K \circ G$ via horizontal diagrammatic composition:



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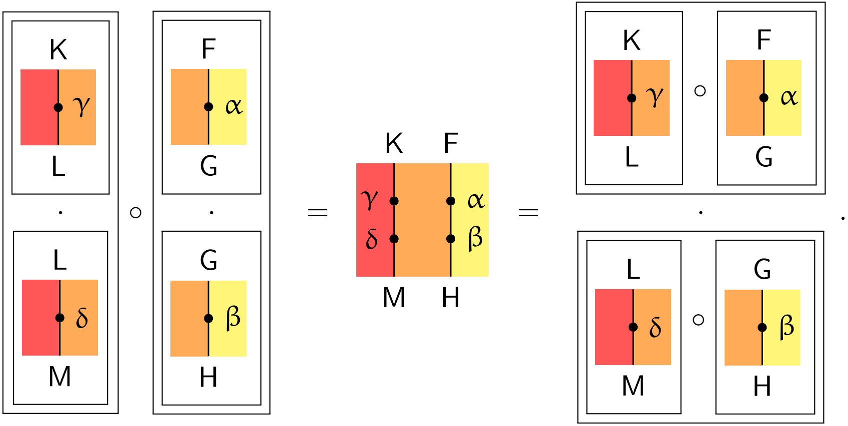
Category Theory in Pictures

The interchange law

The two forms of composition satisfy the **interchange law**:

$$(\delta \cdot \gamma) \circ (\beta \cdot \alpha) = (\delta \circ \beta) \cdot (\gamma \circ \alpha).$$

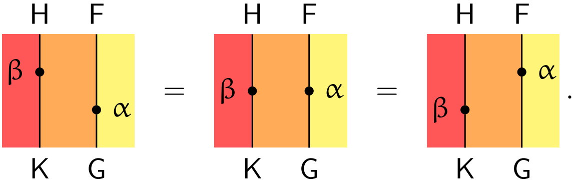
which again is built into the notation:



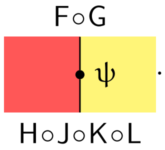
Category Theory in Pictures

Elevator equations

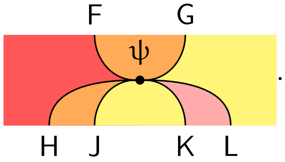
Using both vertical and horizontal composition, and our convention for drawing identity natural transformations, we obtain what Dubuc suggestively calls the **elevator equations**.



We could draw the natural transformation $\psi : F \circ G \xrightarrow{\cdot} H \circ J \circ K \circ L$ as:



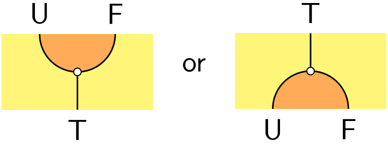
Instead, we draw separate wires for each element of the composite:



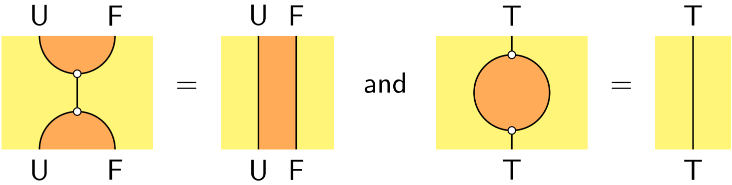
Category Theory in Pictures

Explicit identities and cancellation

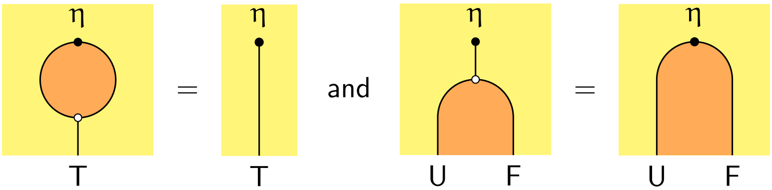
If we have an equation between functors, such as $\mathsf{T} = \mathsf{U} \circ \mathsf{F}$, we can exploit this in our diagrams using explicit identity vertices:



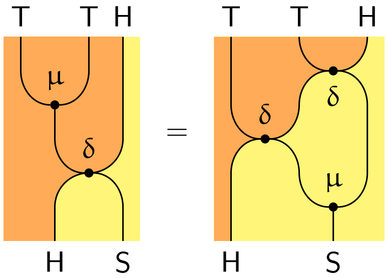
These satisfy obvious cancellation identities:



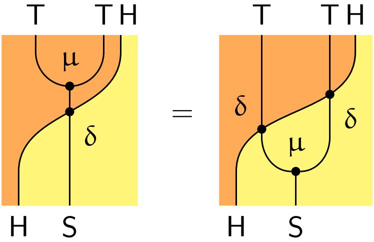
We can also fuse identities with other vertices in obvious ways, for example for $\eta : \mathrm{Id} \dot{\to} T$, with $T = U \circ F$ as before:



Artistic choices matter - overdoing symmetry

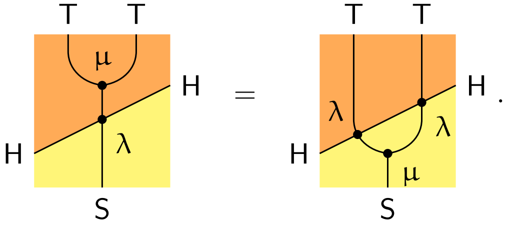


Artistic choices matter - providing intuition



Category Theory in Pictures

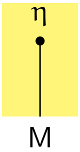
Artistic choices - stretching the notation



Key Structures

Monads

Our fundamental object of study is that of a monad. A **monad** on a category $\mathcal{C}$ consists of an endofunctor $\mathsf{M} : \mathcal{C} \to \mathcal{C}$, and **unit** and **multiplication** natural transformations



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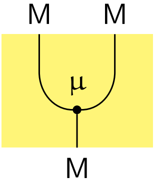

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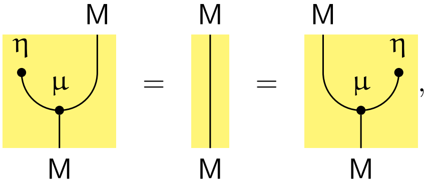
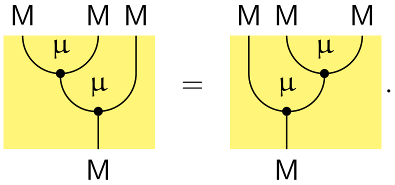
Key Structures

Monad axioms

The unit and multiplication are required to satisfy the following **unitality** and **associativity** equations:

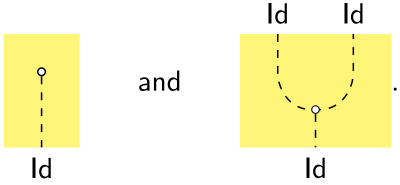




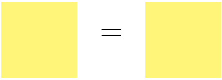
Key Structures

Monads - Example

The identity functor on $\mathcal{C}$ carries the structure of a monad:

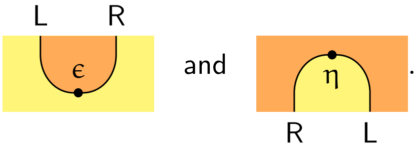


Verifying all the axioms boils down to confirming equations of the form:



Key Structures
Adjunctions

An **adjunction** between a pair of functors $\mathsf{L} : \mathcal{D} \to \mathcal{C}$ and $\mathsf{R} : \mathcal{C} \to \mathcal{D}$ consists of a pair of **counit** and **unit** natural transformations:

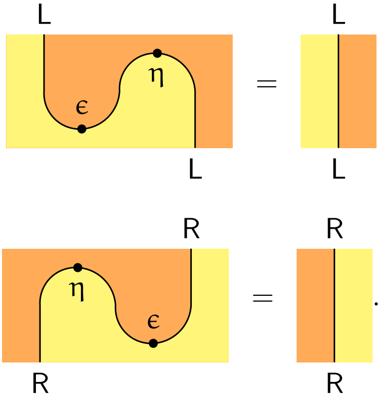


These are often referred to as a **cup** and **cap**.

Key Structures

Adjunction axioms

The unit and counit are required to satisfy the following **snake equations**

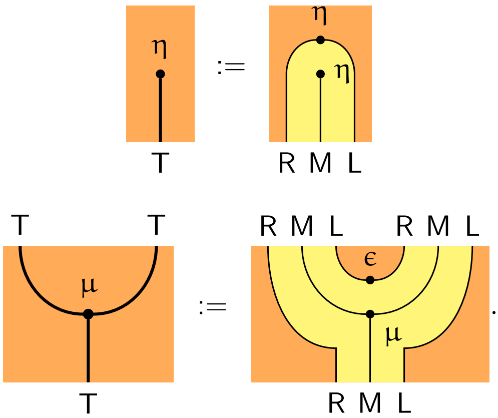


We will write $\mathsf{L} \dashv \mathsf{R} : \mathcal{C} : \mathcal{C} \rightharpoonup \mathcal{D}$ to denote such an adjunction. L and R are referred to as **left** and **right** adjoints respectively.

Key Structures

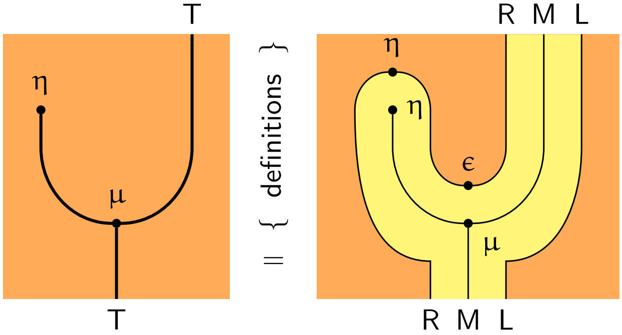
Huber's construction

Given a monad $(\mathsf{M} : \mathcal{C} \to \mathcal{C}, \eta, \mu)$ and an adjunction $\mathsf{L} \dashv \mathsf{R} : \mathcal{C} : \mathcal{C} \rightharpoonup \mathcal{D}$, we can build a new monad on $\mathsf{T} = \mathsf{R} \circ \mathsf{M} \circ \mathsf{L}$ as follows:



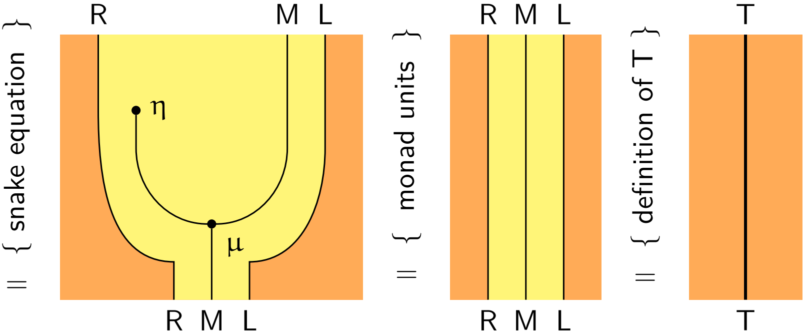
Key Structures

Huber's construction - unit axiom



Key Structures

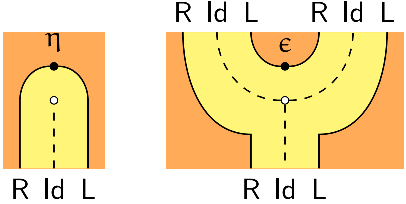
Huber's construction - unit axiom



The other unit and the multiplication axiom have similar proofs.

Key Structures

As a special case of Huber's construction, every adjunction induces a monad:



A **2-category** is an abstraction of the composition of categories, functors and natural transformations, in the same way we can think of categories as an abstraction of how functions between sets compose.

- We now speak of 0,1 and 2-cells.
- There are identity 1 and 2-cells.
- There are two notions of composition, horizontal and vertical.
- Composition and identities satisfy the same equations discussed earlier.

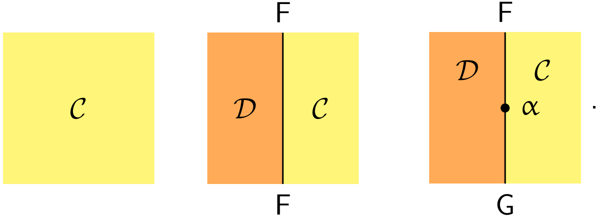
Abstract and Formality

String diagrams as notation for 2-categories

Our graphical notation extends to this abstract setting

Abstract	0-cells	1-cells	2-cells
Concrete	Categories	Functors	Natural Transformations

Graphical



Abstraction and Formality

Transferring notions to other settings

The definitions of monads and adjunctions transfer to the 2-categorical setting, and we can ask what they mean concretely in those settings. For example:

- There is a 2-category in which monads are preordered sets.
- There is a 2-category in which monads are internal categories.
- There is a 2-category in which monads are enriched categories.
- ...

Abstraction and Formality

Formal category theory

We can aim to "do category theory" within an arbitrary 2-category. This is referred to as doing **formal category theory**. Specifically for monads, many results work at this level of abstraction:

- Adjunctions induce monads.
- Monads can be composed using distributive laws.
- Codensity monads are induced by Kan extensions.
- ...

Abstraction and Formality

Monads and adjunctions

Key components of ordinary monad theory:

$$\begin{array}{ccccc} \mathcal{C}_{\mathsf{T}} & \xrightarrow{J} & \mathcal{D} & \xrightarrow{\mathsf{K}} & \mathcal{C}^{\mathsf{T}} \\ \mathsf{F}_{\mathsf{T}} \uparrow \dashv \downarrow \mathsf{U}_{\mathsf{T}} & & \mathsf{F} \uparrow \dashv \downarrow \mathsf{U} & & \mathsf{F}^{\mathsf{T}} \uparrow \dashv \downarrow \mathsf{U}^{\mathsf{T}} \\ C & & C & & C \end{array}$$

Without this, we cannot

- Show every monad arises from an adjunction.
- Show every monad is a codensity monad.
- Even talk about various lifting results.

Abstraction and Formality

The formal theory of monads

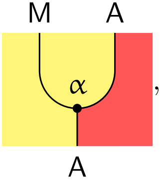
Street's formal theory of monads solves this problem:

- Elegant abstractions of the Eilenberg–Moore and Kleisli constructions are identified.
- The machinery looks rather advanced - auxiliary 2-categories are introduced, and results phrased in terms of the existence of various 2-adjoints.
- A surprisingly large amount of monad theory can be developed at this level of abstraction.

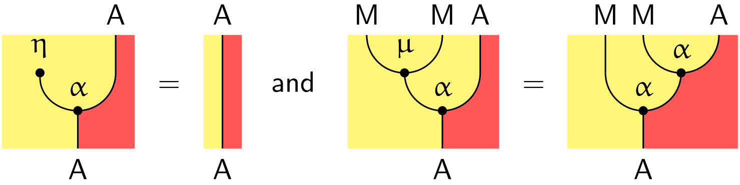
Actions

Left monad actions

Given a monad $(\mathsf{M} : \mathcal{C} \to \mathcal{C}, \eta, \mu)$, a **left** M-**action** of M on $\mathsf{A} : \mathcal{E} \to \mathcal{C}$ is an $\alpha : \mathsf{M} \circ \mathsf{A} \dot{\to} \mathsf{A}$,



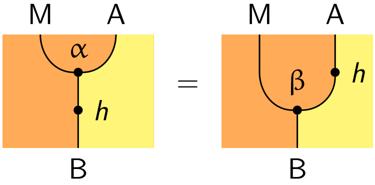
that respects the unit and multiplication, in that the following equations hold.



Actions

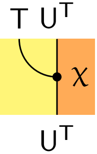
Action transformations

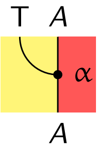
Given a pair of actions of the *same* monad and from the *same* source, a **transformation of actions**, written $h : (\mathsf{A}, \alpha) \to (\mathsf{B}, \beta)$, is a $h : \mathsf{A} \dot{\to} \mathsf{B}$ such that the **right turn axiom** holds:



Actions

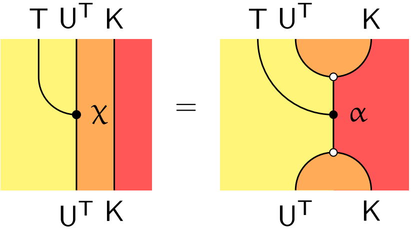
Universal left actions - one dimensional

We say that left T-action 

is **universal** if for every 

there exists a unique 

such that:

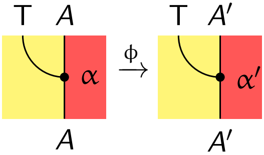


,

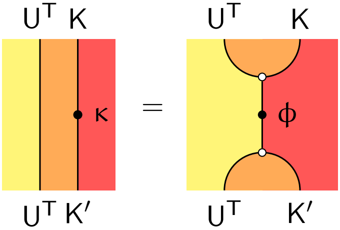
Actions

Universal left actions - two dimensional

...and for every action transformation



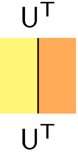
there exists a unique $\kappa : K \dot{\to} K'$ such that

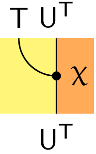


where K and K′ are induced by the actions α and α' respectively.

Actions

Eilenberg–Moore objects

We shall say that  is an **Eilenberg–Moore object** for monad T on if there is a 

carrying a universal left T-action 

.

Symbolically, we shall write $\mathcal{C}^T$ for the Eilenberg–Moore object.

Every monad arises from an adjunction

The plan

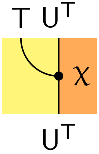
For monad $(\mathsf{T} : \mathcal{C} \to \mathcal{C}, \eta, \mu)$:

- Assume T has an Eilenberg–Moore object.
- Use the universal property to find suitable candidate left and right adjoints.
- Use the universal property to find putative units and counits.
- Show that this data yields an adjunction.
- Show that the adjunction induces the original monad via Huber's construction.

Every monad arises from an adjunction

A candidate right adjoint

Assuming a universal left T-action

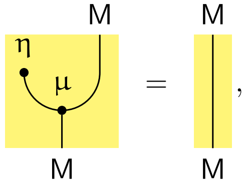


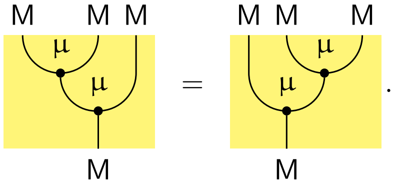
we need to find a candidate right adjoint of type $\mathcal{C}^{\mathsf{T}} \to \mathcal{C}$. We already have a suitable candidate, U^{T}.

Every monad arises from an adjunction

μ is a left T-action

To find a suitable left adjoint, we look for a left T-action that will induce a morphism of the right type. Via the two monad axioms:



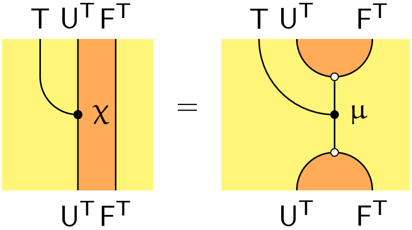


We observe that the multiplication is a left action.

Every monad arises from an adjunction

μ-equation

The universal property tells us that μ will induce $F^T : \mathcal{C} \to \mathcal{C}^T$ such that the following key equation holds:

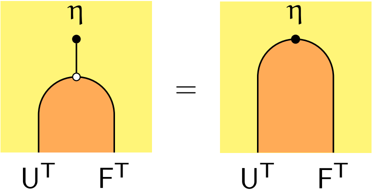


Note in particular that $T = U^T \circ F^T$.

Every monad arises from an adjunction

The inevitable choice of unit

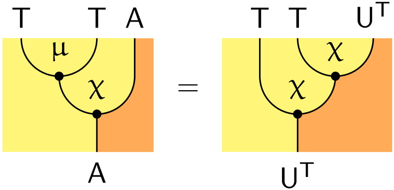
Given the way Huber's construction worked, there is only one choice for the unit of the adjunction, which is to use the unit of the monad, exploiting the fact $T = U^T \circ F^T$.



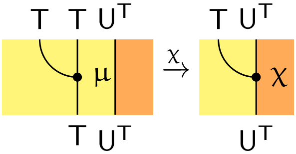
Every monad arises from an adjunction

Constructing a candidate counit

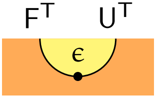
The left T-action axiom



is equivalent to saying χ is an action transformation of type:



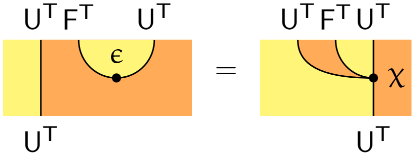
and so by the universal property induces a 2-cell



Every monad arises from an adjunction

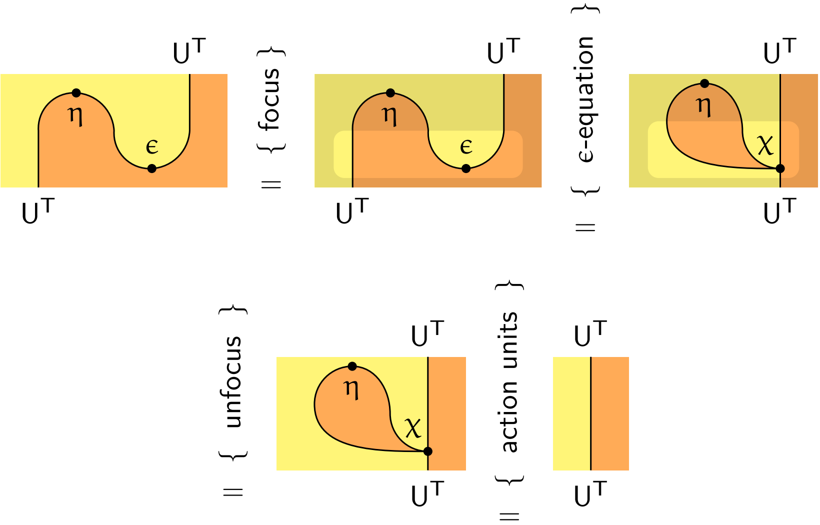
ϵ-equation

Furthermore, the U.P. implies following key equation holds:



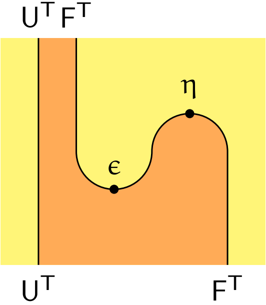
Every monad arises from an adjunction

Proving the first snake equation

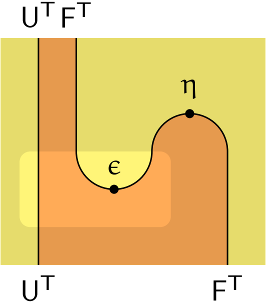


Every monad arises from an adjunction

Proving the second snake equation

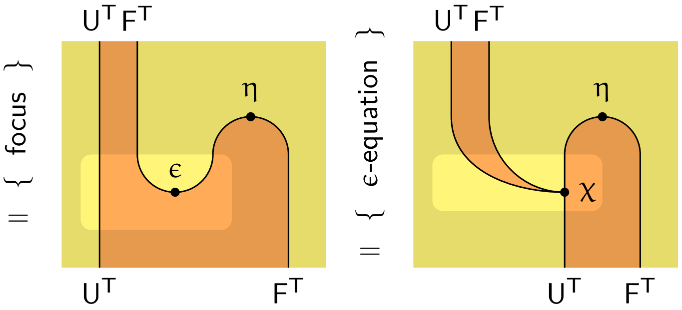


$= \{ \text{ focus } \}$



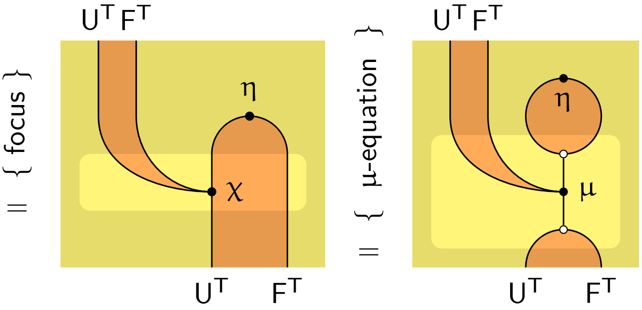
Every monad arises from an adjunction

Proving the second snake equation



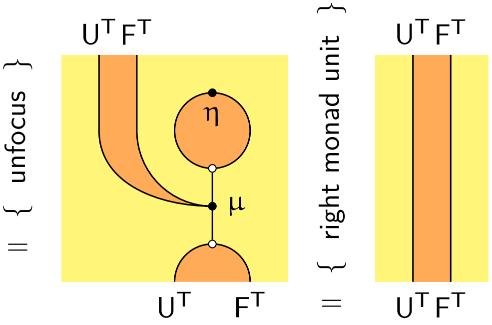
Every monad arises from an adjunction

Proving the second snake equation



Every monad arises from an adjunction

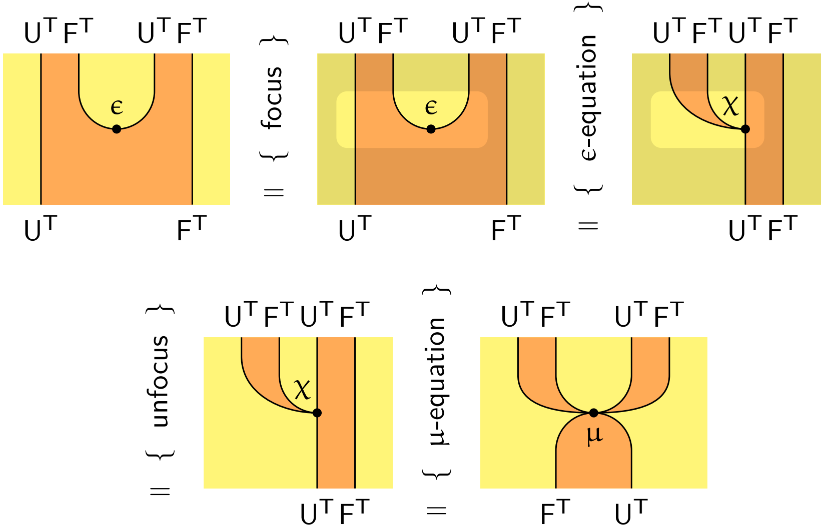
Proving the second snake equation



We then note by the U.P. the operation $U^T \circ (-)$ is injective.

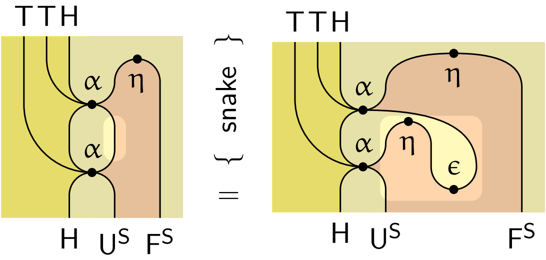
Every monad arises from an adjunction

Confirming we recover the original monad



Conclusion

Going further



- Other results have natural graphical arguments - the Eilenberg–Moore resolution is terminal, Eilenberg–Moore laws classify liftings, Beck distributive laws lift monads to Eilenberg–Moore objects, every monad is a codensity monad,...
- Duality gives results for Kleisli construction and comonads by "flipping pictures'