Relating Structure & Power

... using Game Comonads

Towards generalisations of modal logic

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Outline

- Intro to finite model theory and game comonads
- Applications to modal logic generalisations

What and Why of Finite Model Theory

Finite model theory studies the expressive power of logics on finite models.

- Finite graphs
- Databases

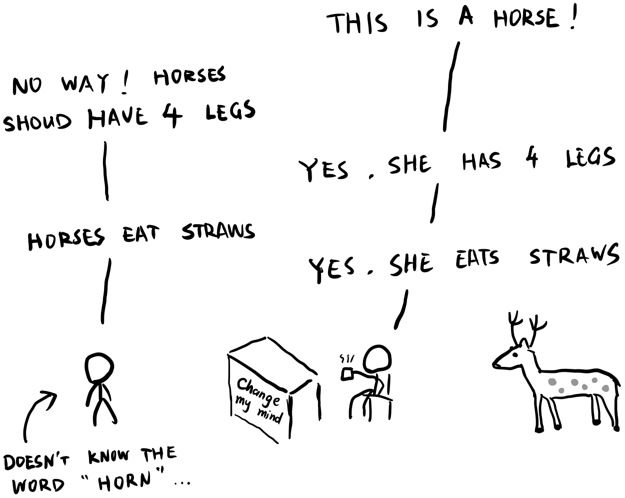
Definition

A relational vocabulary σ is a collection of relation symbols $(P_1, \ldots)$ each with an associated arity.

Definition

A σ-structure $\mathcal{A} = \langle A, \{P_i^{\mathcal{A}}\}\rangle$ consists of a universe A **together with an interpretation** of each k-ary relation symbol as a k-ary relation on A.

Games on Logics



$$\mathcal{A} \equiv^{\mathcal{L}} \mathcal{B} \quad := \quad \forall \phi \in \mathcal{L}.\ \mathcal{A} \models \phi \iff \mathcal{B} \models \phi$$

Example: Ehrenfeucht-Fraïssé Game

Theorem (Existential Ehrenfeucht-Fraïssé).

Let $\mathcal{A}$ and $\mathcal{B}$ be two structures in a relational vocabulary. Then the following are equivalent.

- $\mathcal{A}$ and $\mathcal{B}$ agree on the set of existential positive FO formulae up to quantifier rank k, notated $\exists^+\mathsf{FO}[k]$.
- There is a winning strategy for the duplicator in the k round existential EF game, notated $\mathcal{A} \equiv_k \mathcal{B}$.
- There is a coKleisli morphism $f : \mathbb{E}_k\mathcal{A} \to \mathcal{B}$.

Ehrenfeucht-Fraïssé Games/Comonads [AS21]

- $\mathbb{E}_k$ is a comonad on the category of σ-structures.
- The universe of $\mathbb{E}_k\mathcal{A}$ is the set of spoiler plays, i.e. non-empty sequences of elements of A.
- The counit $\epsilon_{\mathcal{A}}$ returns the last move of a play.
- $R^{\mathbb{E}_k\mathcal{A}}(s_1, \ldots, s_n) \iff$

$$\forall i, j \in [n].(s_i \sqsubseteq s_j \vee s_j \sqsubseteq s_i) \wedge R^{\mathcal{A}}(\epsilon(s_1, \ldots, \epsilon(s_n))$$

- For $f \colon \mathbb{E}_k\mathcal{A} \to \mathcal{B}$, the coextension $f^* \colon \mathbb{E}_k\mathcal{A} \to \mathbb{E}_k\mathcal{A}$ is defined recursively as

$$f^*(s\#[a]) := f^*(s)\#[f([a])]$$

EF Games/Comonads cont.[AS21]

- $f\colon \mathbb{E}_k\mathcal{A} \to \mathcal{B}$ encodes the duplicator response to a given spoiler play, i.e. a duplicator strategy.
- f preserving relations implies that the duplicator strategy is *winning*.
- Coextension $f^*\colon \mathbb{E}_k\mathcal{A} \to \mathbb{E}_k\mathcal{B}$ models history preservation of the game.

Coalgebras of Game Comonads

Why do we care about game comonads?

- Capture multiple model comparison games with a single abstraction.

The coalgebras of the $\mathbb{E}_k$ / $\mathbb{P}_k$ / $\mathbb{M}_k$ correspond to ...

- the **tree-depth** / **tree-width** / **synchronisation tree depth**
- ... of a tree cover
- ... of the Gaifman graph of $\mathcal{G}(\mathcal{A})$

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- Applications to modal logic generalisations

Restricting First Order Logic [Var96]

- FOL is great but its SAT problem is undecidable.
- The SAT problem of Modal logic is *robustly decidable*, but lacks expressive power.
- We can view modal logic as a fragment of FOL.

Can we extend modal logic without losing SAT?
YES!

- Unary-negation fragment of FO
- Ordered fragments of FO

The UNFO fragment [SC13]

- $\phi, \psi := x \mid R(\vec{x}) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \exists x.\phi \mid x_1 = x_2 \mid \neg\phi(x)$.
- We can only negate when there's one single free variable.
- We can only write $\forall$ for one single free variable.
- We can **not** write $x \neq y$!

$$\neg\exists y, z, u.E(x, y) \wedge E(y, z) \wedge E(z, u) \wedge R(u, x) \quad \checkmark$$
$$\neg\exists x.R(x, y, z) \quad \times$$

The UNFO game [SC13]

Let us have two structures, A and B. We have a starting position, $(a, b) = (\perp, \perp)$.

1. Spoiler chooses a structure. Say A.
2. Spoiler chooses a subset $V \subseteq A$.
3. Duplicator gives a partial homomorphism $h : A \to B$ defined on V, such that $h(a) = b$.
4. Spoiler chooses an element $a' \in V$, and fixes a new $(a, b) = (a', h(a'))$.

Duplicator wins the k round if they can give such a partial homomorphism.

An equivalent game: the Pebble UNFO game

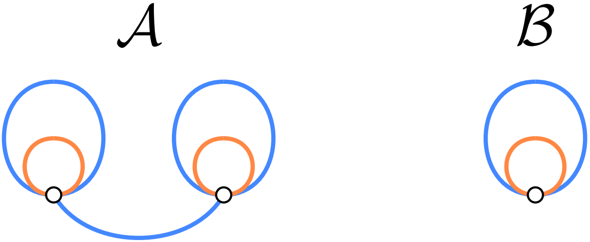
Let us have two structures, $\mathcal{A}$ and $\mathcal{B}$.

- Spoiler chooses a structure. As long as they stay in that one, the game goes like in the EF case.
- When the Spoiler changes structure, they choose a play (a, b) and forget the rest. The game then continues like the EF case.

Duplicator wins the round if the current board forms a partial homomorphism.

- We can write a variant where instead of choosing a play (a, b) we just keep the last play.
- These two versions are equivalent to the original UNFO game.

Smiley



$$\exists x.\ \exists y.\ B(x,y)\ \wedge\ \neg R(x,y)$$

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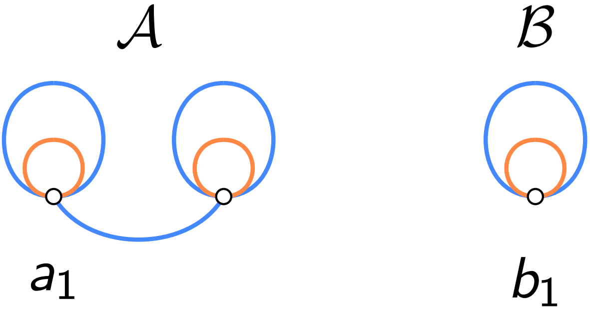

$$\exists x.\, \exists y.\, B(x, y) \;\wedge\; \neg R(x, y)$$

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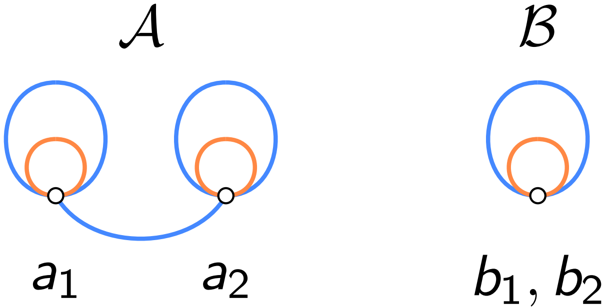

$$\exists x.\, \exists y.\, B(x, y) \;\wedge\; \neg R(x, y)$$

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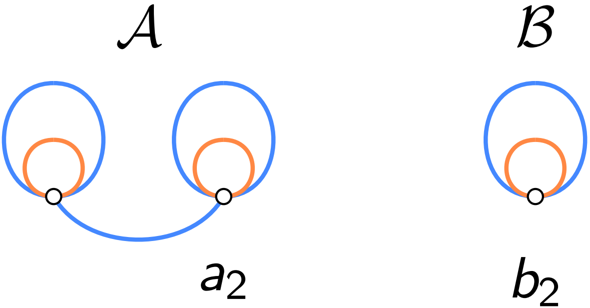


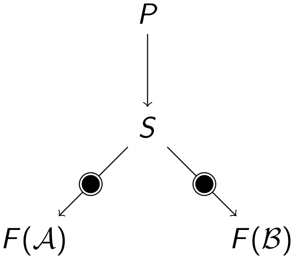
$$\exists x.\, \exists y.\, B(x,y) \ \wedge\ \neg R(x,y)$$

Span of Open Pathwise Embeddings

Theorem [AS21]

The existence of a span of open pathwise embeddings in the EM($\mathbb{E}_k$) corresponds to

- a duplicator winning strategy in the *back and forth* EF-game.
- $\mathcal{A} \equiv^{FO} \mathcal{B}$.



Extending to Pebble-UNFO Game

Ordered fragments of First Order logic [BJ22]

- Variables can only appear in the order they are introduced.
- **Prefix fragment**: using only prefixes. (PL)
- **Infix fragment**: any infix is valid. (IL)
- **Fluted fragment**: using only postfixes. (FL)

	PL	**IL**	**FL**
$\exists x_1, x_2, x_3, R(x_1, x_2)$	✓	✓	×
$\exists x_1, x_2, x_3, R(x_2)$	×	✓	×
$\exists x_1, x_2, x_3, R(x_2, x_3)$	×	✓	✓
$\exists x_1, x_2, x_3, R(x_3, x_2)$	×	×	×
$\exists x_1, x_2, x_3, R(x_1, x_1)$	×	×	×

Comonadification of Ordered Fragments

- We found comonads for all 3 fragments!

The coalgebras of the comonad correspond to ...

- the **directed forest height**
- ... of a forest cover
- ... of the Gaifman graph of $\mathcal{G}(\mathcal{A})$

Ongoing Work

- Extend ordered fragment comonads to allow variable rebinding.
- Formalize coherence conditions for span representation of UNFO game.

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Category of σ-structures

Definition

Given a relational vocabulary σ, the category $\mathcal{R}(\sigma)$ has

- objects are σ-structures,
- a morphism $\mathcal{A} \to \mathcal{B}$ is a structure homomorphism, i.e. a set-function $h\colon A \to B$ that preserves relations:

$$R^{\mathcal{A}}(a_1, \ldots, a_n) \implies R^{\mathcal{B}}(h(a_1), \ldots, h(a_n)) \tag{1}$$

  for all $R \in \sigma$.

Chapter 3: Ehrenfeucht-Fraïssé Games

The Ehrenfeucht-Fraïssé game is a two-player sequential move game with the following components.
Players:

- Spoiler
- Duplicator

Board: two structures, e.g. $\mathfrak{A}$ and $\mathfrak{B}$
Goal:

- Spoiler wants to show that the two structures are different
- Duplicator wants to show that the two structures are the same

Chapter 3: Ehrenfeucht-Fraïssé Games

How to play:

- The players play a certain number of rounds.
- In each round, the spoiler picks a structure $\mathfrak{A}$ or $\mathfrak{B}$ and an element of that structure $a \in \mathfrak{A}$ or $b \in \mathfrak{B}$.
- The duplicator responds by picking an element from the other structure.

Winning:

- Let $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$ be the moves played after n rounds of an E-F Game. Also, let $\vec{c}^{\mathfrak{A}}$ denote $(c_1^{\mathfrak{A}}, \ldots, c_l^{\mathfrak{A}})$ and similarly for $\vec{c}^{\mathfrak{B}}$.
- $(\vec{a}, \vec{b})$ is a winning position for the duplicator if $((\vec{a}, \vec{c}^{\mathfrak{A}}), (\vec{b}, \vec{c}^{\mathfrak{B}}))$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.
- When the duplicator has an n-round winning strategy, write $\mathfrak{A} \equiv_n \mathfrak{B}$.