Adjoint School Research Direction

Nice generalisations of modal logic

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Modal logic

Syntax

$$\varphi = p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond\varphi$$

Semantics via FO

Semantics via the **standard translation** to a unary formula in first-order logic:

$$\begin{aligned}
[\![p]\!]_x &= P(x) \\
[\![\varphi \wedge \psi]\!]_x &= [\![\varphi]\!]_x \wedge [\![\psi]\!]_x \\
[\![\neg\varphi]\!]_x &= \neg[\![\varphi]\!]_x \\
[\![\Diamond\varphi]\!]_x &= \exists y.\, E(x, y) \wedge [\![\varphi]\!]_y
\end{aligned}$$

High level attributes of modal logic

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- Is computationally very well behaved.
- Is model theoretically very well behaved.
- These properties are robust to extensions such as backwards or global modalities.
- Lacks expressive power.

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Famous Paper

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Finding nice generalisations of the modal fragment

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- Quantification only leaves one variable free.
- All negations are only applied to unary formulae.
- Variables in atoms are only used in very restricted ways.

Guarded quantification

Guarded logic

The **atom guarded quantification fragment** only permits quantification of the form:

$$\exists \overline{y}.\alpha(\overline{xy}) \wedge \varphi(\overline{z})$$

where $\overline{z}$ is some subsets of the variables in $\overline{xy}$, and α is some atom. For example:

$\exists y.\ E(y, x) \wedge \varphi(y)$	backward modalities
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Categorical semantics

Abramsky and Marsden. "Comonadic semantics for guarded fragments", 2021.

Guarded negation

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An even more expressive fragment is the **(atom) guarded negation fragment**, which restricts negations to the form:

$$\alpha(\overline{xy}) \wedge \neg\varphi(\overline{y})$$

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Categorical semantics

To be worked out! Challenges:

- More challenging multi-phase games.
- Existential positive fragment is unrestricted.

Variable count restrictions

Two-variable fragment

The **two-variable fragment** FO2 simply restricts the number of variables that can be used in formulae to two.

$$\varphi(x) \equiv \exists y.\, E(x,y) \land \exists x.\, E(y,x) \land \exists y.\, E(x,y) \land P(y)$$

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Categorical semantics

- Captured by the pebbling comonad from the original game comonad paper (Abramsky, Dawar, and Wang. "The pebbling comonad in Finite Model Theory", 2017).
- Challenge - Conceptual proof FO2 has the finite model property.

Variable use order restrictions

Fluted, forward and ordered fragment

The **fluted, forward and ordered fragments** can only use certain ordered sequences of quantified variables, for example:

$\exists x_1. \forall x_2. \exists x_3. R(x_1, x_2, x_3) \wedge E(x_2, x_3)$	fluted (suffix)
$\exists x_1. \forall x_2. \exists x_3. R(x_1, x_2, x_3) \wedge E(x_1, x_2)$	ordered (prefix)
$\exists x_1. \forall x_2. \exists x_3. R(x_1, x_2, x_3) \wedge P(x_2)$	forward (subsequence)
$\exists x_1. \forall x_2. \exists x_3. R(x_1, x_2, x_3) \wedge E(x_1, x_1)$	non example
$\exists x_1. \forall x_2. \exists x_3. R(x_1, x_2, x_3) \wedge E(x_1, x_3)$	non example

(Bednarczyk and Jaakkola. "Towards a Model Theory of Ordered Logics: Expressivity and Interpolation", 2022).

Variable use order restrictions

Adjacent Fragment

The **adjacent fragment** (Bednarczyk, Kojelis, and Pratt-Hartmann. "On the Limits of Decision: the Adjacent Fragment of First-Order Logic", 2023). generalises the previous fragment, now allowing any adjacent sequence of variables in a relational atom. For example:

$\exists x_1.\ \forall x_2.\ \exists x_3.\ S(x_1, x_2, x_1, x_2, x_2, x_3)$	adjacent
$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge E(x_2, x_3)$	adjacent
$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge E(x_1, x_2)$	adjacent
$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge P(x_2)$	adjacent
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$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge E(x_1, x_3)$	still non example

Variable use order restrictions

Categorical semantics

To be worked out!

- Unlike other fragments considered so far.
- Can be mixed in with other features like guards to get better combined properties - see the reference for ordered logics.
- These fragments, particularly the recent adjacent fragment are at first sight rather puzzling - what's going on?

Quantification restriction - take two

One-dimensional fragment

The **one dimensional fragment** (Kieronski and Kuusisto. "Uniform One-Dimensional Fragments with One Equivalence Relation", 2015) only allows blocks of quantification of the form:

$$\psi(x) \equiv \exists \overline{x}.\ \varphi$$

leaving at most only one free variable.

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leaving at most only one free variable. In fact, the well-behaved fragment is the **uniform one-dimensional fragment** which imposes a uniformity condition on variables in Boolean combinations of relational atoms.

Categorical semantics

To be worked out!

- Unlike other fragments considered so far.
- Another unusual model comparison game structure.

Suggested Plan of Attack

- Start with the unary negation fragment - already non-trivial but avoids some of the difficulties of the guarded negation fragment. Potentially head towards GNFO if successful.

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- Consider the other fragment types, such ordered variable and one-dimensional fragments, and how to incorporate their game structures.
- Ideal outcome - A publishable categorical account of some or all of these fragments, including new categorical model theory developments, and clarification of the more traditional work.
- Minimum outcome - We all learn a lot about practically relevant fragments of first-order logic, and the challenges involved in studying them from a structural point of view.

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