

String Diagrams for Elementary Category Theory

2: Monads

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Based on joint work with Ralf Hinze

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Monads

Defined in old school notation

Definition (Monad)

A **monad** on category \mathcal{C} consists of a functor $M : \mathcal{C} \rightarrow \mathcal{C}$ and natural transformations:

$$\eta : \text{Id} \xrightarrow{\cdot} M,$$

$$\mu : M \circ M \xrightarrow{\cdot} M.$$

satisfying **unit** and **associativity** axioms:

$$\mu \cdot (\eta \circ M) = id = \mu \cdot (M \circ \eta),$$

$$\mu \cdot (\mu \circ M) = \mu \cdot (M \circ \mu).$$

Monads

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$$\begin{aligned}\eta : \text{Id} &\rightarrow M, \\ \mu : M \circ M &\rightarrow M.\end{aligned}$$

satisfying **unit** and **associativity** axioms:

$$\begin{array}{ccc} M & \xrightarrow{\eta \circ M} & M \circ M & \xleftarrow{M \circ \eta} & M \\ & \searrow id_M & \downarrow \mu & \swarrow id_M & \\ & & M & & \end{array} \quad \begin{array}{ccc} M \circ M \circ M & \xrightarrow{\mu \circ M} & M \circ M & \xrightarrow{\mu} & M \\ M \circ \mu \downarrow & & & & \downarrow \mu \\ M \circ M & \xrightarrow{\mu} & M & & \end{array}$$

Monads

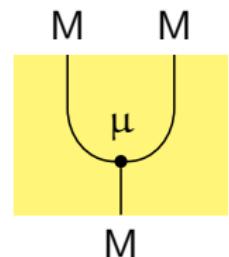
Defined using string diagrams

Definition (Monad)

A monad on category \mathcal{C} consists of a functor $M : \mathcal{C} \rightarrow \mathcal{C}$ and natural transformations:



and



Monads

Defined using string diagrams

Definition (Monad)

satisfying the **unit** equations:

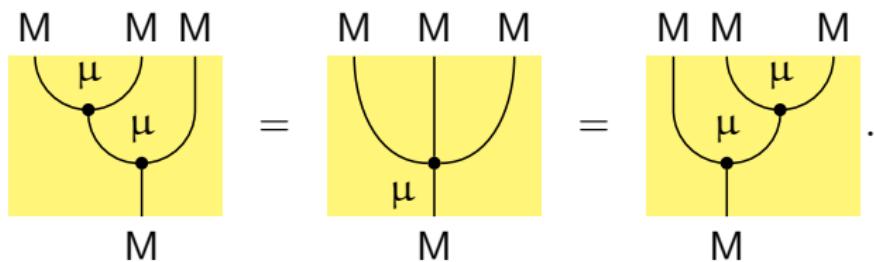
$$\begin{array}{c} M \\ \eta \bullet \mu \\ \hline M \end{array} = \begin{array}{c} M \\ | \\ M \end{array} = \begin{array}{c} M \\ \mu \bullet \eta \\ \hline M \end{array},$$

Monads

Defined using string diagrams

Definition (Monad)

and the **associativity** equation:

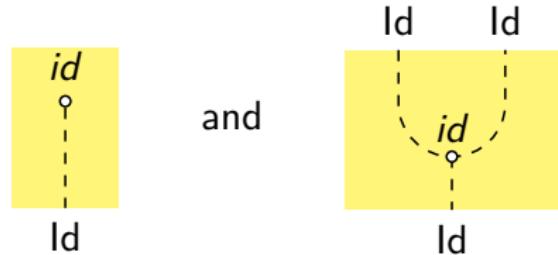


Monads

Examples

Example (The identity monad)

The identity monad on \mathcal{C} consists of the identity functor $\text{Id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ and unit and multiplication both identity natural transformations:



Monads

Examples

Example (The list monad)

The functor $L : \mathbf{Set} \rightarrow \mathbf{Set}$ mapping each set X to the set of lists of elements from X is a monad.

- ▶ The unit at X , $\eta_X : X \rightarrow L(X)$ maps an element $x \in X$ to the singleton list $[x]$.
- ▶ The multiplication at X , $\mu : L(L(X)) \rightarrow L(X)$ maps a list of lists to a list by concatenating all its elements. For example:

$$[[1, 2, 3], [4, 5], [6]] \mapsto [1, 2, 3, 4, 5, 6]$$

Monads

Examples

Example (The multiset monad)

Let the functor $M : \mathbf{Set} \rightarrow \mathbf{Set}$ map each set X to the set of *finite* multisets (or bags) from X . For example:

$$\{a : 2, b : 1\}$$

This functor is a monad.

- ▶ The unit at X , $\eta X : X \rightarrow M(X)$ maps an element $x \in X$ to the singleton multiset $\{x : 1\}$.
- ▶ The multiplication at X , $\mu X : M(M(X)) \rightarrow M(X)$ maps a multiset of multisets to a multiset by taking “unions” which account for multiplicities. For example:

$$\{\{a : 1, b : 2\} : 2, \{a : 3, c : 1\} : 1\} \mapsto \{a : 5, b : 4, c : 1\}$$

Monad morphisms

Definition (Monad morphism)

Given a pair of monads over the *same* base category, $S, T : \mathcal{C} \rightarrow \mathcal{C}$, a monad morphism is a natural transformation $\tau : S \rightarrow T$ such that:

$$\tau \cdot \eta = \eta \quad \text{and} \quad \tau \cdot \mu = \mu \cdot (\tau \circ \tau)$$

Monad morphisms

Definition (Monad morphism)

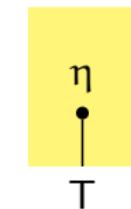
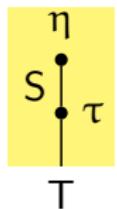
Given a pair of monads over the *same* base category, $S, T : \mathcal{C} \rightarrow \mathcal{C}$, a monad morphism is a natural transformation $\tau : S \rightarrow T$ such that:

$$\begin{array}{ccc} \text{Id}_{\mathcal{C}} & \xrightarrow{\eta} & S \\ & \searrow \eta & \downarrow \tau \\ & & T \end{array} \quad \text{and} \quad \begin{array}{ccc} S \circ S & \xrightarrow{\tau \circ \tau} & T \circ T \\ \mu \downarrow & & \downarrow \mu \\ S & \xrightarrow{\tau} & T \end{array}$$

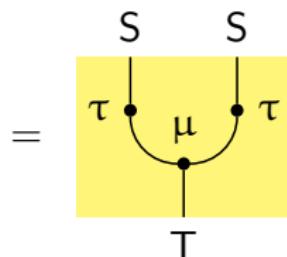
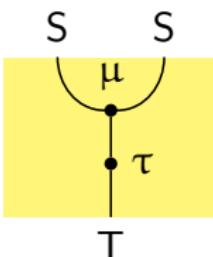
Monad morphisms

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and



Monad morphisms

Composition

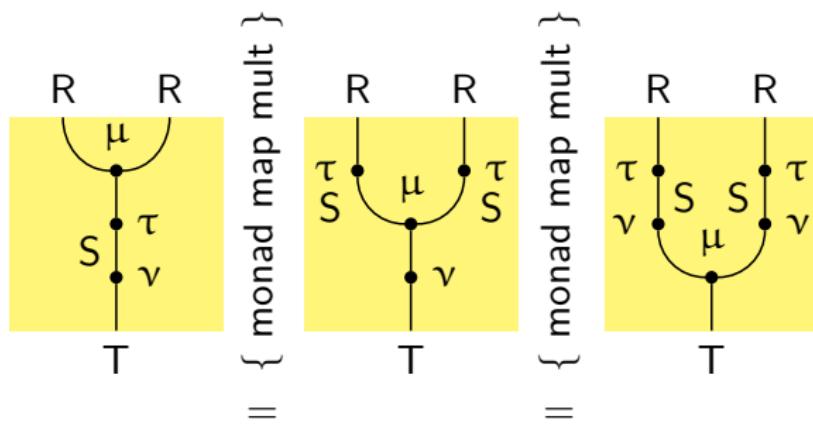
Unit preservation:

$$\begin{array}{c}
 \text{Diagram 1: } \text{S} \xrightarrow{\gamma} \text{R} \xrightarrow{\eta} \text{T} \\
 \text{Diagram 2: } \text{S} \xrightarrow{\gamma} \text{T} \\
 \text{Diagram 3: } \text{T}
 \end{array}
 \quad \text{||} \quad \left\{ \begin{array}{c} \text{monad map unit } \eta \\ \text{monad map unit } \gamma \end{array} \right\} \quad \text{,}$$

Monad morphisms

Composition

Multiplication preservation:



Monad morphisms

Examples

Example (Monad units)

$$\begin{array}{c} \eta \\ \circ \\ \bullet \quad \eta \\ | \\ M \end{array} = \begin{array}{c} \eta \\ \bullet \\ | \\ M \end{array}$$

$$\begin{array}{c} \text{Id} \quad \text{Id} \\ \backslash \quad / \\ \circ \quad \circ \\ | \quad | \\ \bullet \quad \eta \\ | \\ M \end{array} = \begin{array}{c} \text{Id} \quad \text{Id} \\ \backslash \quad / \\ \eta \quad \bullet \quad \mu \quad \bullet \quad \eta \\ \curvearrowleft \quad \curvearrowright \\ | \quad | \\ \bullet \quad \eta \\ | \\ M \end{array}.$$

Monad morphisms

Examples

Example (Lists to multisets)

There is a monad morphism $L \rightarrow M$ from the list to the multiset monad, mapping a list to the multiset of elements that appear, with their multiplicities. For example:

$$[a, b, a, c] \mapsto \{a : 2, b : 1, c : 1\}$$

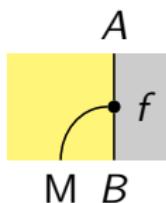
The Kleisli category of a monad

Definition (Kleisli category, part I)

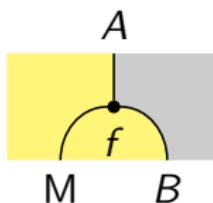
For a monad (M, η, μ) , the **Kleisli category**, denoted \mathcal{C}_M , has the same objects as \mathcal{C} but the arrows differ. A Kleisli arrow $A \rightarrow B : \mathcal{C}_M$ is an arrow of type $A \rightarrow M B : \mathcal{C}$ in the underlying category.

The Kleisli category of a monad

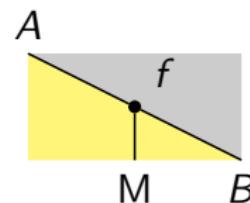
Some options for drawing Kleisli morphisms:



versus



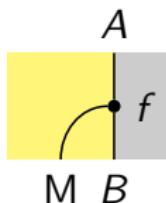
versus



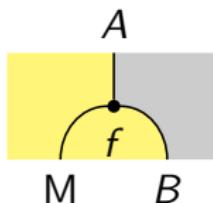
.

The Kleisli category of a monad

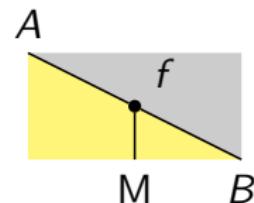
Some options for drawing Kleisli morphisms:



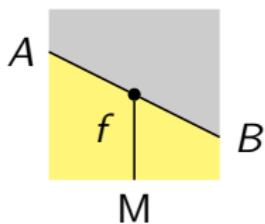
versus



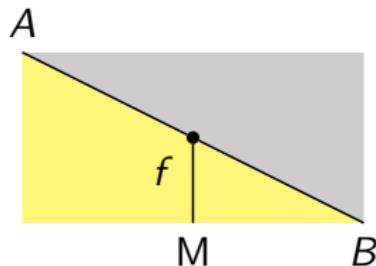
versus



Compact notation:



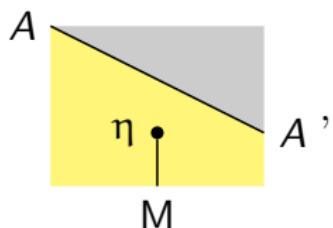
extends to



The Kleisli category of a monad

Definition (Kleisli category, part II)

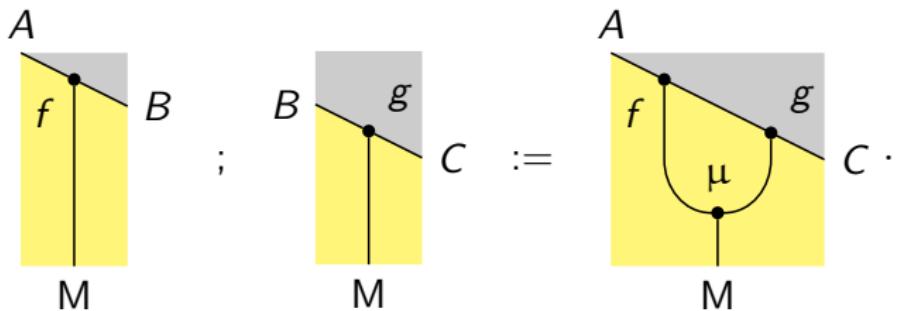
Identity on A in \mathcal{C}_M is given by $\eta A : A \rightarrow MA : \mathcal{C}$. Graphically:



The Kleisli category of a monad

Definition (Kleisli category, part III)

Composition in \mathcal{C}_M is defined as:



The Kleisli category for a monad

Checking the category axioms

Identity on the left:

$$\begin{array}{c}
 \text{id}_A ; f \\
 \parallel \{ \text{definition} \} \\
 \text{A} \quad \text{B} \\
 \eta \quad f \\
 \mu \\
 \text{M} \\
 \text{B} , \\
 \text{id}_A ; f \\
 \parallel \{ \text{monad unit} \} \\
 \text{A} \quad \text{B} \\
 f \\
 \text{M} \\
 \text{B} ,
 \end{array}$$

The Kleisli category for a monad

Checking the category axioms

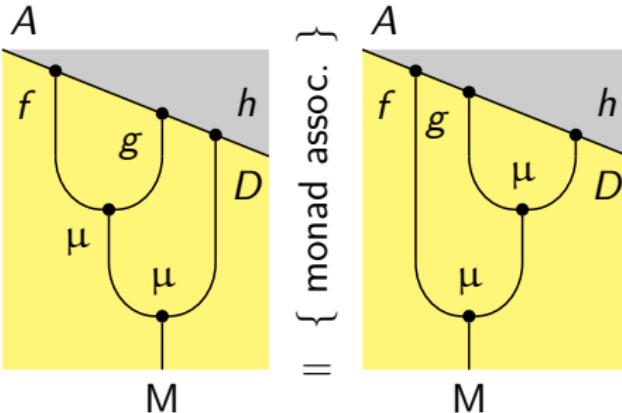
Identity on the right:

$$\begin{array}{c} f ; id_B \\ = \{ \text{definition} \} \\ \text{---} \\ \begin{array}{c} A \\ \diagdown \quad \diagup \\ \text{---} \\ f \quad \eta \\ \diagup \quad \diagdown \\ \text{---} \\ \mu \\ \text{---} \\ M \end{array} \end{array} \quad \begin{array}{c} A \\ \diagdown \quad \diagup \\ \text{---} \\ f \\ \text{---} \\ M \end{array} = \{ \text{monad unit} \} \quad \begin{array}{c} B \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ B \end{array} .$$

The Kleisli category for a monad

Checking the category axioms

Associativity of composition:

$$(f ; g) ; h = \{ \text{monad assoc.} \} = f ; (g ; h)$$


Examples

Example (Identity monad)

The Kleisli category of the identity monad $\text{Id} : \mathbf{Set} \rightarrow \mathbf{Set}$ is simply the category of sets and functions.

Examples

Example (List monad)

The Kleisli category of the list monad $L : \mathbf{Set} \rightarrow \mathbf{Set}$ has morphisms the form:

$$X \rightarrow LY$$

We can think of this as a nondeterministic computation, mapping an element of X to a list of possible outcomes in Y .

Examples

Example (Multiset monad)

The Kleisli category of the multiset monad $M : \mathbf{Set} \rightarrow \mathbf{Set}$ has morphisms the form:

$$X \rightarrow MY$$

We can think of this as a quantified nondeterministic computation, mapping an element of X to a multiset of possible outcomes in Y , tracking the multiplicity of each occurrence.

Algebras for an endofunctor

Definition (F-algebra)

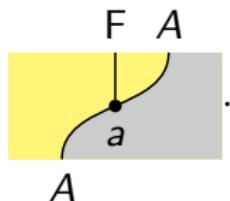
Given an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$, an **F-algebra** is a pair (A, a) , where A is an object in \mathcal{C} , and $a : FA \rightarrow A$ is an arrow in \mathcal{C} . These are known as the **carrier** and **action** of the algebra respectively.

Algebras for an endofunctor

Definition (F-algebra)

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Graphically:



Algebras for an endofunctor

Definition (F-algebra homomorphism)

An **F-homomorphism** between algebras (A, a) and (B, b) is an arrow $h : A \rightarrow B$ in the underlying category \mathcal{C} such that the **homomorphism axiom** holds:

$$h \cdot a = b \cdot F h.$$

Algebras for an endofunctor

Definition (F-algebra homomorphism)

An **F-homomorphism** between algebras (A, a) and (B, b) is an arrow $h : A \rightarrow B$ in the underlying category \mathcal{C} such that the **homomorphism axiom** holds:

$$h \cdot a = b \cdot F h.$$

Graphically this condition is:

$$\begin{array}{ccc} F & A \\ \text{---} & \text{---} \\ h & a \\ \text{---} & \text{---} \\ B & & \end{array} = \begin{array}{ccc} F & & A \\ \text{---} & \text{---} & \text{---} \\ b & & h \\ \text{---} & \text{---} & \text{---} \\ B & & \end{array}.$$

F-algebras form a category

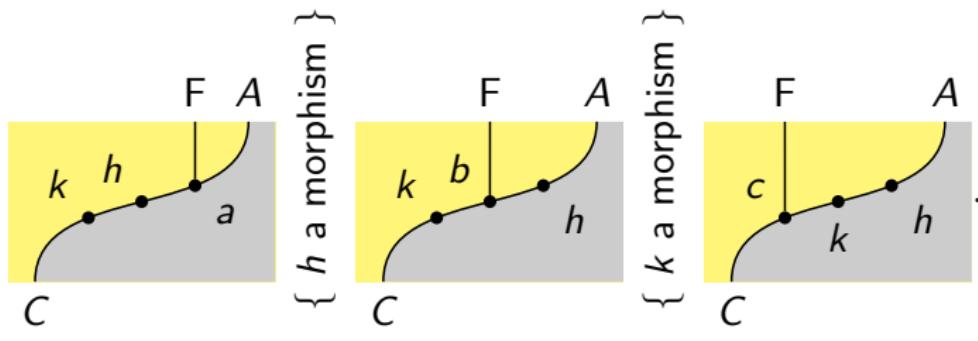
The identity morphisms in \mathcal{C} are F-algebra morphisms. To show to show $a \cdot F id = id \cdot a$ graphically:

$$\begin{array}{c} F \ A \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} F \ A \\ \text{---} \\ \text{---} \end{array}.$$

The diagram consists of two identical F-algebra boxes side-by-side, separated by an equals sign. Each box has a yellow top section labeled $F \ A$ and a grey bottom section labeled A . A curved arrow labeled a points from the bottom section to the top section. A vertical line segment connects the two sections.

F-algebras form a category

F-algebra morphisms compose:



F-algebras form a category

Therefore, for $F : \mathcal{C} \rightarrow \mathcal{C}$, F-algebras and their homomorphisms form a category $F\text{-}\mathbf{Alg}(\mathcal{C})$, with composition and identities as in \mathcal{C} .

Algebras for a monad

Definition (Eilenberg–Moore algebra)

Given a monad $M : \mathcal{C} \rightarrow \mathcal{C}$, an **algebra for M** , also referred to as an **Eilenberg–Moore algebra for M** , is an M -algebra (A, a) that satisfies the **unit** and **multiplication** axioms, or coherence conditions.

$$a \cdot \eta A = id \quad \text{and} \quad a \cdot \mu A = a \cdot M a.$$

Algebras for a monad

Definition (Eilenberg–Moore algebra)

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Graphically the unit axiom is:

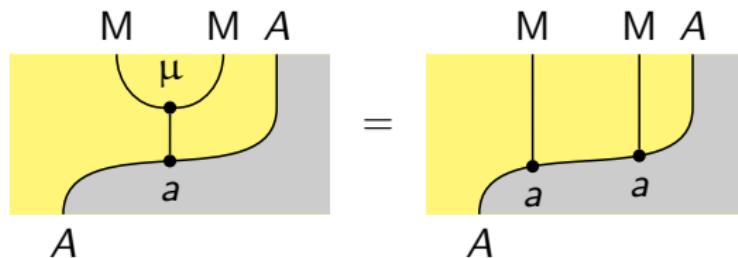
The diagram illustrates the unit axiom for an M -algebra (A, a) . It consists of two parts separated by an equals sign ($=$). The left part shows a yellow square representing the carrier set A , with a grey curved shape representing the monad M . A point labeled a is on the boundary of the grey shape. A point labeled η is inside the yellow square. A vertical line segment connects η to a . The right part shows a yellow square representing the carrier set A , with a grey curved shape representing the monad M . The boundary of the grey shape is identical to the one in the left part, but there is no point labeled a on it.

Algebras for a monad

Definition (Eilenberg–Moore algebra)

Given a monad $M : \mathcal{C} \rightarrow \mathcal{C}$, an **algebra for M** , also referred to as an **Eilenberg–Moore algebra for M** , is an M -algebra (A, a) that satisfies the **unit** and **multiplication** axioms, or coherence conditions.

Graphically the multiplication axiom is:



Algebras for a monad

Definition (Eilenberg–Moore algebra)

Given a monad $M : \mathcal{C} \rightarrow \mathcal{C}$, an **algebra for M** , also referred to as an **Eilenberg–Moore algebra for M** , is an M -algebra (A, a) that satisfies the **unit** and **multiplication** axioms, or coherence conditions.

The category of M -algebras and M -homomorphisms is known as the **Eilenberg–Moore category** of M , denoted \mathcal{C}^M .

Algebras for a monad

Examples

Example (Identity monad)

The Eilenberg–Moore category of the identity monad

$\text{Id} : \mathbf{Set} \rightarrow \mathbf{Set}$ is equivalent (in fact isomorphic) to the category \mathbf{Set} .

Algebras for a monad

Examples

Example (List monad)

The Eilenberg–Moore category of the list monad $L : \mathbf{Set} \rightarrow \mathbf{Set}$ is equivalent (in fact isomorphic) to the category of monoids and monoid homomorphisms.

Algebras for a monad

Examples

Example (Multiset monad)

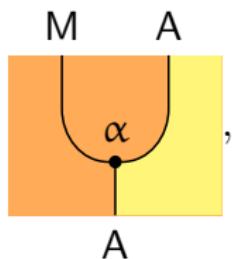
The Eilenberg–Moore category of the multiset monad

$M : \mathbf{Set} \rightarrow \mathbf{Set}$ is equivalent (in fact isomorphic) to the category of commutative monoids and monoid homomorphisms.

Monad actions

Definition (Left action of a monad)

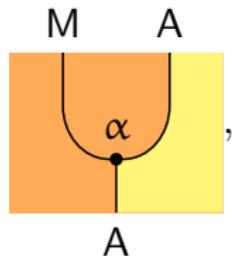
Given a monad (M, η, μ) on a category \mathcal{C} , a **left action of M** on a functor $A : \mathcal{C} \rightarrow \mathcal{D}$ is a natural transformation $\alpha : M \circ A \rightarrow A$:



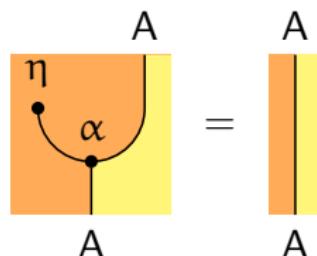
Monad actions

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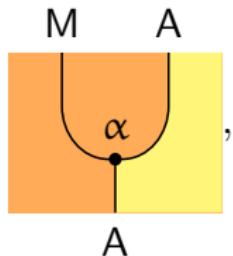
α must respect the unit:



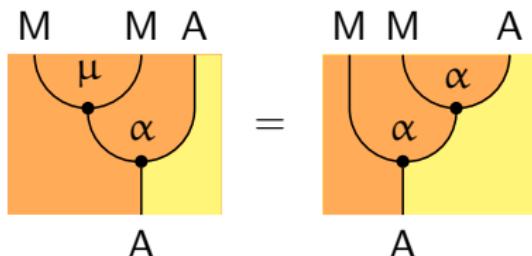
Monad actions

Definition (Left action of a monad)

Given a monad (M, η, μ) on a category \mathcal{C} , a **left action of M** on a functor $A : \mathcal{C} \rightarrow \mathcal{D}$ is a natural transformation $\alpha : M \circ A \rightarrow A$:



and multiplication:

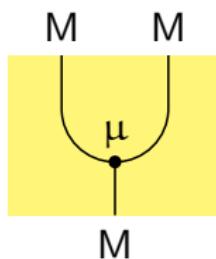


Monad actions

Examples

Example (Multiplication)

For any monad M , the multiplication μ is a (left) action of the monad M on the endofunctor M .

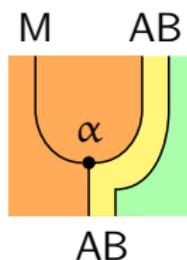


Monad actions

Examples

Example (Outlining on the right)

Given an action $\alpha : M \circ A \rightarrow A$, we can form new actions by outlining the right edge as follows.

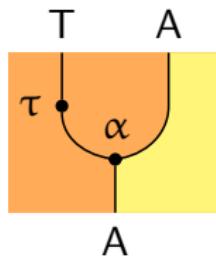


Monad actions

Examples

Example (Dots on the left)

Given an action $\alpha : M \circ A \rightarrow A$, and a monad (T, η, μ) , we can form new actions by placing a dot of type $T \rightarrow M$ on the left prong. We then consider what axioms τ should satisfy.



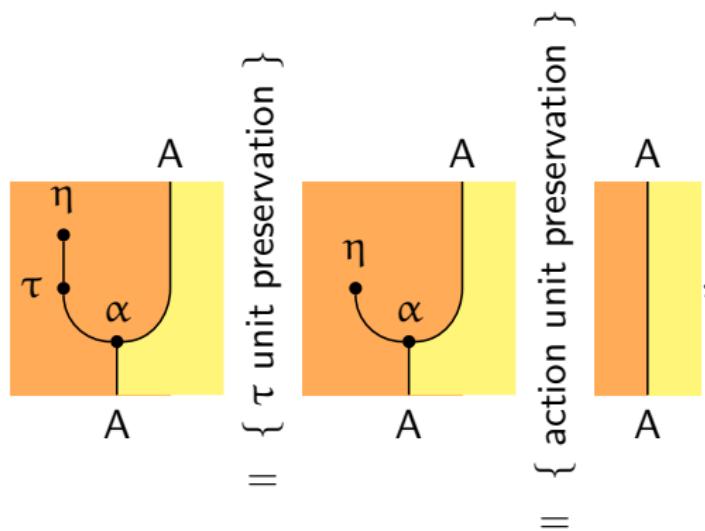
Monad actions

Examples

Example (Dots on the left)

Given an action $\alpha : M \circ A \xrightarrow{\cdot} A$, and a monad (T, η, μ) , we can form new actions by placing a dot of type $T \xrightarrow{\cdot} M$ on the left prong. We then consider what axioms τ should satisfy.

Checking the action unit preservation:



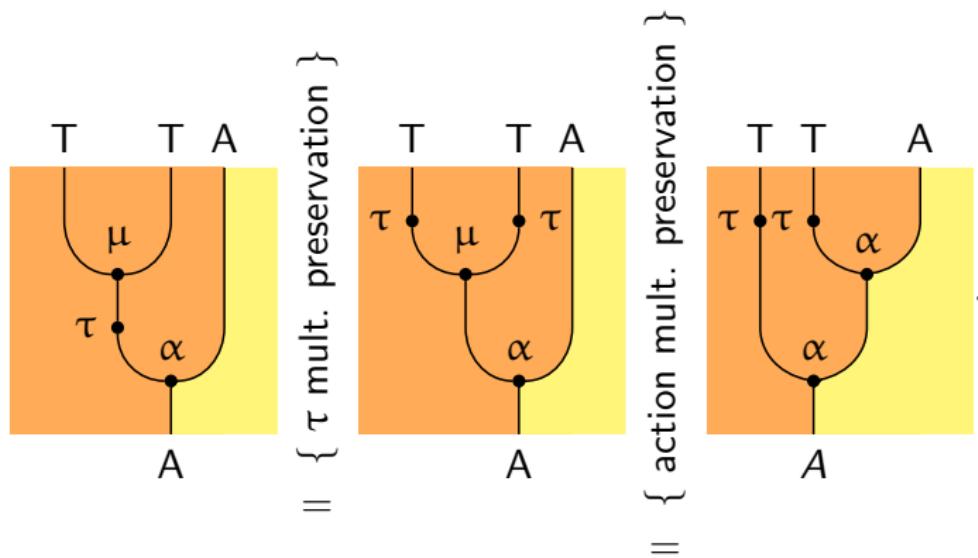
Monad actions

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Checking the action multiplication preservation:



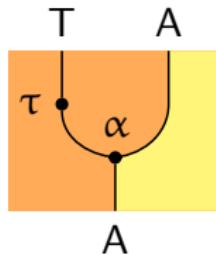
Monad actions

Examples

Example (Dots on the left)

Given an action $\alpha : M \circ A \xrightarrow{\cdot} A$, and a monad (T, η, μ) , we can form new actions by placing a dot of type $T \xrightarrow{\cdot} M$ on the left prong. We then consider what axioms τ should satisfy.

Hence, if τ is a monad morphism, then the composite

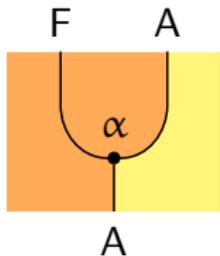


is a valid monad action.

Endofunctor actions

Definition (Vanilla action)

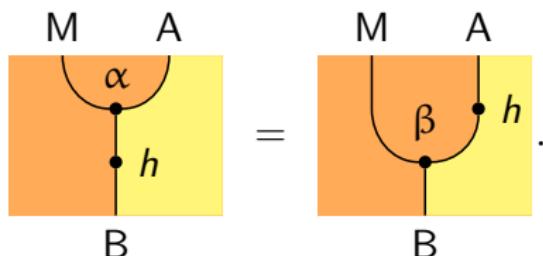
Given an endofunctor F on a category \mathcal{C} , a **left action of F** on a functor $A : \mathcal{B} \rightarrow \mathcal{C}$ is a natural transformation $\alpha : F \circ A \rightarrow A$. (Not required to satisfy any additional axioms). In pictures, it is any α of the form:



Transformations of actions

Definition (Transformations of actions)

A **transformation of actions**, written $h : (A, \alpha) \rightarrow (B, \beta)$, is a natural transformation $h : A \rightarrow B$ such that the **right turn axiom** holds:



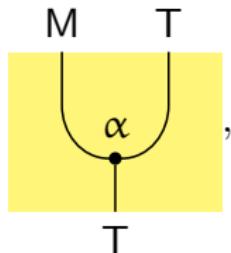
Transformations of vanilla actions are defined analogously.

Relating algebras and actions

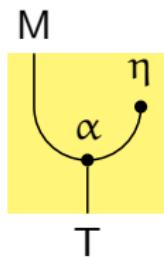
F -algebra	vanilla F -action/natural F -algebra
$(A : \mathcal{C}, a : FA \rightarrow A)$	$(A : \mathcal{B} \rightarrow \mathcal{C}, \alpha : F \circ A \rightarrow A)$
algebra for M	left action of M /natural algebra for M
$(A : \mathcal{C}, a : MA \rightarrow A)$	$(A : \mathcal{B} \rightarrow \mathcal{C}, \alpha : M \circ A \rightarrow A)$
homomorphism	transformation of actions/natural homomorphism
$h : (A, a) \rightarrow (B, b)$	$h : (A, \alpha) \rightarrow (B, \beta)$

Compatible actions

Given a monad action:



where (T, η, μ) is a monad, when is the composite:



a monad morphism?

Compatible actions

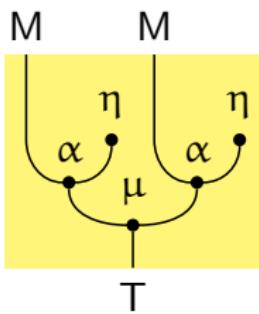
Checking the monad morphism unit preservation axiom:

$$\text{unit } \eta \circ \alpha = \text{unit } \eta$$

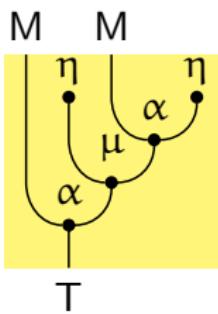
The diagram illustrates the unit preservation axiom for a monad morphism. It consists of two parts separated by an equals sign (=).
The left part shows a yellow square containing a commutative diagram. It has two horizontal lines. The top line has two black dots labeled η and a curved arrow labeled α connecting them. The bottom line has a single black dot labeled η . The two lines meet at a central black dot, which is connected to the bottom line by a vertical line labeled $\text{unit } \eta$.
The right part shows a yellow square containing a single vertical line labeled $\text{unit } \eta$, representing the right-hand side of the equation.

Compatible actions

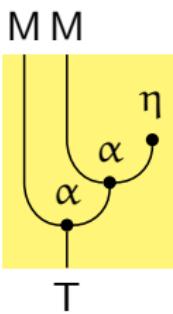
Attempting the monad morphism multiplication preservation axiom:



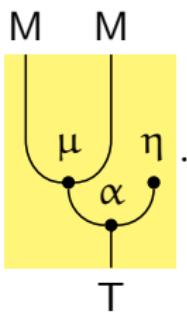
$\Downarrow \{ \text{unjustified step} \}$



$\Downarrow \{ \text{monad unit axiom} \}$



$\Downarrow \{ \text{action mult. axiom} \}$



Compatible actions

From the previous calculation we have discovered the following **pseudo-associativity** axiom is sufficient:

$$\begin{array}{c} M \quad T \quad T \\ \alpha \quad \mu \\ \text{---} \\ T \end{array} = \begin{array}{c} M \quad T \quad T \\ \mu \quad \alpha \\ \text{---} \\ T \end{array}.$$

We shall call such an $\alpha : M \circ T \rightarrow T$ a **compatible action**.

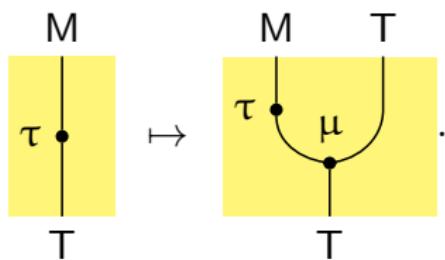
Monad morphisms to compatible actions

Given a monad map $\tau : M \rightarrow T$, can we construct a compatible action?

Monad morphisms to compatible actions

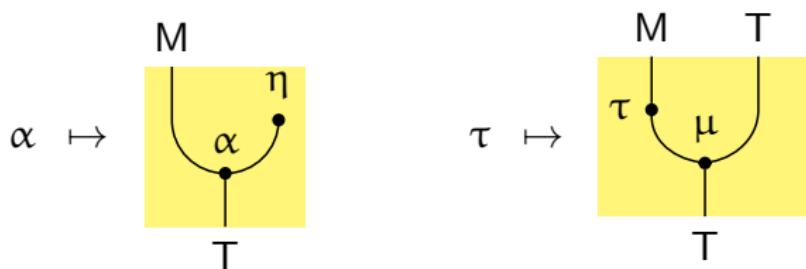
Given a monad map $\tau : M \rightarrow T$, can we construct a compatible action?

Yes! Via the mapping:



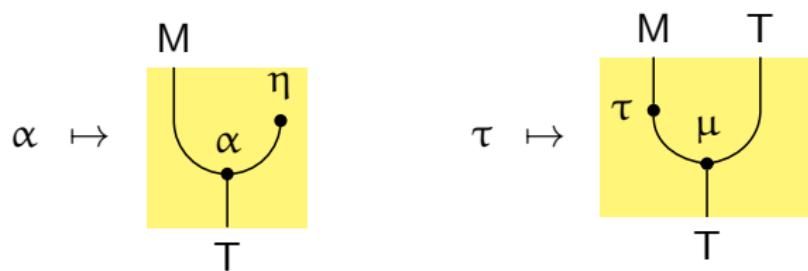
Monad morphisms to compatible actions

In fact, we have a bijection between M -actions compatible with T and monad morphisms $M \rightarrow T$:



Monad morphisms to compatible actions

In fact, we have a bijection between M -actions compatible with T and monad morphisms $M \xrightarrow{\cdot} T$:



In an entirely analogous manner, we can establish a bijection between vanilla actions $F \circ T \rightarrow T$ compatible with T , and natural transformations $F \xrightarrow{\cdot} T$.

Next time

Adjunctions!